Lecture 23
The jamming game

This work is licensed under the Creative Commons Attribution-ShareAlike 4.0 International License.
Lecture 23— Contents

Jamming as a mutual information game
  Introduction
  The MI game on an AWGN channel

MI jamming game over parallel AWGN channels
The problem of communication jamming

- A denial of service attack
- Any device with antenna is a potential jammer
- Traditionally protect transmissions at the physical layer
- Wideband / narrowband, spreading / hopping solutions
Formulation as a decoding game

\[ w \sim \mathcal{N}(0, \sigma^2) \]

**Player Strategies, Objective, and Constraints**

<table>
<thead>
<tr>
<th>Player</th>
<th>Strategies</th>
<th>Objective</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A, B)</td>
<td>(Tx, Rc, ( p_u ))</td>
<td>( \max R_u )</td>
<td>( P[\hat{u} \neq u] &lt; \varepsilon )</td>
</tr>
<tr>
<td>J</td>
<td>( p_v ) or ( p_v</td>
<td>z )</td>
<td>( \max P[\hat{u} \neq u] )</td>
</tr>
</tbody>
</table>

Nicola Laurenti

The jamming game

December 12, 2017
Formulation as a mutual information game

By the Channel Coding Theorem

If there exists $p_x$ such that $R_u < I(x; y)$, then there exist $T_x$ and $R_c$ such that $P[\hat{u} \neq u] < \varepsilon$, $\forall \varepsilon > 0$

<table>
<thead>
<tr>
<th>player</th>
<th>strategy</th>
<th>objective</th>
<th>constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$p_x$</td>
<td>$\max I(x; y)$</td>
<td>$\mathbb{E}[x^2] \leq P_T$</td>
</tr>
<tr>
<td>J</td>
<td>$p_v$ or $p_v</td>
<td>z$</td>
<td>$\min I(x; y)$</td>
</tr>
</tbody>
</table>
Purely adversarial games

A game of two players, A vs J, with strategies \( s_A \in S_A, s_J \in S_J \), respectively, where

- A aims at \textbf{maximizing} \( f(s_A, s_J) \)
- J aims at \textbf{minimizing} \( f(s_A, s_J) \) (equiv., maximizing \(-f(s_A, s_J)\))

is called \textit{purely adversarial} or \textit{zero-sum}.

The \textbf{best response} for A is a function

\[
\hat{s}_A : S_J \mapsto S_A, \quad \hat{s}_A(s_J) = \arg \max_{s_A} f(s_A, s_J)
\]

Similarly, the \textbf{best response} for J is a function

\[
\hat{s}_J : S_A \mapsto S_J, \quad \hat{s}_J(s_A) = \arg \min_{s_J} f(s_A, s_J)
\]
Minimax solutions and Nash equilibrium

In a purely adversarial game, the minimax solution for $A$ is the pair $(s_A^*, s_J^*)$ where

$$s_A^* = \arg\max_{s_A} f(s_A, \hat{s}_J(s_A)) , \quad s_J^* = \hat{s}_J(s_A^*)$$

Similarly, the minimax solution for $J$ is the pair $(s_A^0, s_J^0)$ where

$$s_J^0 = \arg\min_{s_J} f(\hat{s}_A(s_J), s_J) , \quad s_A^0 = \hat{s}_A(s_J^0)$$

If the two minimax solutions coincide, they are also a Nash equilibrium for the game, called the saddle point equilibrium.
Mutual info game on an AWGN channel: formulation

\[ w \sim \mathcal{N}(0, \sigma^2) \]

\[ y = g_A x + g_J v + w = \tilde{x} + \tilde{v} \]

The utility function can be written as

\[ I(x; y) = I(\tilde{x}; y) = \mathbb{E} \left[ \log_2 \frac{p_y(\tilde{x}|y)}{p_y(y)} \right] = \mathbb{E} \left[ \log_2 \frac{p_\tilde{v}(\tilde{v})}{p_y(y)} \right] \]

since \( p_y(\tilde{x}|y) = p_\tilde{v}(y - \tilde{x}) = p_\tilde{v}(\tilde{v}) \)
Kullback-Leibler divergence

**Definition**

Given two rvs, $x, y$ with the same alphabet $\mathcal{A}$ and pdfs $p_x, p_y$ with support $\sigma(p_x) \subset \sigma(p_y) \subset \mathcal{A}$, their Kullback-Leibler divergence is

$$D (p_x \| p_y) = \mathbb{E} \left[ \log_2 \frac{p_x(x)}{p_y(x)} \right] = \int_{\sigma(p_x)} p_x(a) \log_2 \frac{p_x(a)}{p_y(a)} \, da$$

**Properties**

1. **(asymmetry)** $D (p_x \| p_y) \neq D (p_y \| p_x)$, in general
2. **(positivity)** $D (p_x \| p_y) \geq 0$, \forall p_x, p_y
   
   and $D (p_x \| p_y) = 0$ if and only if $p_x \equiv p_y$
3. **(relation with entropy)** If $x, y$ are discrete and $y \sim \mathcal{U}(\mathcal{A})$,
   
   $D (p_x \| p_y) = \log_2 |\mathcal{A}| = H(x)$
Kullback-Leibler divergence (cont.)

Properties (cont.)

4. **(Gaussian rvs)** If $x, y$ are continuous and $y \sim \mathcal{N}(0, M_y)$,

$$D(p_x \| p_y) = \frac{1 + \log_2 \pi}{2} + E \left[ \log_2 (\sigma_y p_x(x)) \right] + \frac{1}{2 \ln 2} \frac{M_x}{\sigma_y^2}$$

Then, if $x'$ is any rv with $M_{x'} = M_x$ as $x$ (e.g., $x' \sim \mathcal{N}(0, M_x)$)

$$E \left[ \log_2 \frac{p_x(x)}{p_y(x')} \right] = D(p_x \| p_y)$$

5. **(data processing inequality)** If $\tilde{x} = f(x, z)$ and $\tilde{y} = f(y, z)$ for some function $f$ and rv $z$, then $D(p_x \| p_y) \geq D(p_{\tilde{x}} \| p_{\tilde{y}})$

\[
\begin{array}{c}
  x \rightarrow f(\cdot, z) \rightarrow \tilde{x} \\
  \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
   \quad \quad \quad \quad \quad \quad \\
  y \rightarrow f(\cdot, z) \rightarrow \tilde{y}
\end{array}
\]
Best signal with Gaussian jamming

Proposition

If \( v \sim \mathcal{N}(0, M_v) \), then the best response for A is \( \hat{x}(v) \sim \mathcal{N}(0, P_T) \).

Proof.

Since \( v \sim \mathcal{N}(0, M_v) \), then \( \tilde{v} \sim \mathcal{N}(0, M_{\tilde{v}}) \).

Also, for any \( x, y \), let \( x_G \sim \mathcal{N}(0, M_x) \), \( y_G = g_A x_G + \tilde{v} \sim \mathcal{N}(0, M_y) \).

We will show that \( I(x; y) \leq I(x_G; y_G) \). In fact,

\[
I(x_G; y_G) - I(x; y) = E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_y(y)} \right] - E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_y(y)} \right]
\]

\[
= E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_y(y)} \right] = E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_{\tilde{v}}(\tilde{v})} \right]
\]

\[
= E \left[ \log_2 \frac{p_y(y)}{p_y(y)} \right] = D(p_y \parallel p_{\tilde{v}}) \geq 0
\]

Since \( I(x_G; y_G) = \frac{1}{2} \log_2(1 + M_{\tilde{x}}/M_{\tilde{v}}) \) is an increasing function of \( M_x \), the maximum is achieved by a Gaussian \( x \) with maximum power.
Worst jamming with Gaussian signal

Proposition

If $x \sim \mathcal{N}(0, M_x)$, then the best response for $J$ is $\hat{v}(x) \sim \mathcal{N}(0, P_J)$

Proof.

Since $x \sim \mathcal{N}(0, M_x)$, then $\tilde{x} \sim \mathcal{N}(0, M_{\tilde{x}})$.

Also, for any $v, y$, let $v_G \sim \mathcal{N}(0, M_v), y_G = g_J v_G + w + \tilde{x} \sim \mathcal{N}(0, M_y)$.

We will show that $I(x; y) \geq I(x; y_G)$ In fact,

$$I(x; y) - I(x; y_G) = E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_y(y)} \right] - E \left[ \log_2 \frac{p_{\tilde{v}_G}(\tilde{v}_G)}{p_{y_G}(y_G)} \right]$$

$$= E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_y(y)} \frac{p_{\tilde{v}_G}(\tilde{v}_G)}{p_{y_G}(y_G)} \right]$$

$$= E \left[ \log_2 \frac{p_{\tilde{v}}(\tilde{v})}{p_{\tilde{v}_G}(\tilde{v}_G)} \right] - E \left[ \log_2 \frac{p_y(y)}{p_{y_G}(y_G)} \right]$$
Worst jamming with Gaussian signal

Proof (cont.)

\[
I(x; y) - I(x; y_G) = E \left[ \log_2 \frac{p_{\tilde{y}}(\tilde{v})}{p_{\tilde{v}_G}(\tilde{v})} \right] - E \left[ \log_2 \frac{p_y(y)}{p_{y_G}(y)} \right] \\
= D(p_{\tilde{v}} \| p_{\tilde{v}_G}) - D(p_y \| p_{y_G})
\]

Note that \( y \) and \( y_G \) are the outputs of the jamming channel, with \( v \) and \( v_G \) as corresponding inputs.
Thus, by the data processing inequality, \( D(p_y \| p_{y_G}) \leq D(p_{\tilde{v}} \| p_{\tilde{v}_G}) \), and we have proved that \( I(x; y_G) \leq I(x; y) \).

Finally, since \( I(x; y_G) = \frac{1}{2} \log_2(1 + M_{\tilde{x}}/M_{\tilde{v}}) \) is a decreasing function of \( M_v \), the minimum is achieved by a Gaussian \( v \) with maximum power. \( \square \)
The saddle point solution

The above propositions show that

- \( v^* \sim \mathcal{N}(0, P_J) \) is the best response to any zero mean Gaussian signal
- \( x^* \sim \mathcal{N}(0, P_T) \) is the best response to any zero mean Gaussian jamming

In particular, \( x^*, v^* \) are the best response to each other. Therefore, they represent the minimax solution for both A and J, and the saddle point Nash equilibrium of the game.
MI jamming game over parallel AWGN channels

Consider that A, B and J have access to \( L \) parallel identical AWGN channels.

- With independent noises \( w_1, \ldots, w_L \) independent inputs \( x_i \) and \( v_i \) are optimal
- The Nash equilibrium on each channel \( i = 1, \ldots, L \) is zero mean Gaussian signal and jamming.

Then, the strategy for A and J is how to allocate the transmitted powers \( M_{x_i}, M_{v_i} \) over the channels

<table>
<thead>
<tr>
<th>player</th>
<th>strategy</th>
<th>objective</th>
<th>constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( M_{x_1}, \ldots, M_{x_L} )</td>
<td>( \max \sum_i I(x_i; y_i) )</td>
<td>( \sum_i M_{x_i} \leq P_T )</td>
</tr>
<tr>
<td>J</td>
<td>( M_{v_1}, \ldots, M_{v_L} )</td>
<td>( \min \sum_i I(x_i; y_i) )</td>
<td>( \sum_i M_{v_i} \leq P_J )</td>
</tr>
</tbody>
</table>
Jensen inequalities

Let $g(\cdot)$ be a concave (or a convex) function over an interval $\mathcal{I} \subset \mathbb{R}$. The following inequalities hold

**Finite average form**

For all $N$, and for all choices of $a_i \in \mathcal{I}, i = 1, \ldots, N$

\[
\frac{1}{N} \sum_i g(a_i) \leq g \left( \frac{1}{N} \sum_i a_i \right) \quad \text{if } g \text{ is concave}
\]

\[
\frac{1}{N} \sum_i g(a_i) \geq g \left( \frac{1}{N} \sum_i a_i \right) \quad \text{if } g \text{ is convex}
\]

**Statistical mean form**

For any rv $z$ taking values in $\mathcal{I}$

\[
E[g(z)] \leq g \left( E[z] \right) \quad \text{if } g \text{ is concave}
\]

\[
E[g(z)] \geq g \left( E[z] \right) \quad \text{if } g \text{ is convex}
\]
Pure strategy Nash equilibrium

The utility function can be written as

$$\sum_{i=1}^{L} f(M_{x_i}, M_{v_i}), \quad f(M_x, M_v) = \frac{1}{2} \log_2 \left( 1 + \frac{g_A M_x}{\sigma^2 + g_J M_v} \right)$$

where $f(M_x, M_v)$ is increasing and concave wrt $M_x$, and is decreasing and convex wrt $M_v$

- by the increasing/decreasing property the best responses must have
  $\sum_i M_x^* = P_T$ and $\sum_i M_v^* = P_J$
- by the finite average Jensen inequality
  $$\sum_i f(M_{x_i}, \overline{M_v}) \leq L f(\overline{M_x}, \overline{M_v}) \leq \sum_i f(\overline{M_x}, M_{v_i})$$
  where averages are denoted as $\overline{M_x} = \frac{1}{L} \sum_i M_{x_i}$ and $\overline{M_v} = \frac{1}{L} \sum_i M_{v_i}$
- then the saddle point of the pure strategy game is given by
  $M_{x_i}^* = P_T / L, M_{v_i}^* = P_J / L$, for all $i = 1, \ldots, L$

Thus, the Nash equilibrium is for both A and J to equally distribute their available power over all channels
Is random hopping over channels beneficial?

Let us allow $M_{x_i}, M_{v_i}$ to be (nonnegative) random variables (mixed strategy), with the total expected power constraints

$$\sum_i E[M_{x_i}] \leq P_T, \quad \sum_i E[M_{v_i}] \leq P_J$$

and consider the expected utility function

$$\sum_i I(x; y) = \sum_i E[f(M_{x_i}, M_{v_i})]$$

Then, by applying the Jensen inequality, first in the statistical expectation form and then in the finite average form

$$\sum_i E[f(M_{x_i}, M_{v_i}^*)] \leq \sum_i f(E[M_{x_i}], M_{v_i}^*) \leq \sum_i f(M_{x_i}^*, M_{v_i}^*)$$

$$\sum_i E[f(M_{x_i}^*, M_{v_i})] \geq \sum_i f(M_{x_i}^*, E[M_{v_i}]) \geq \sum_i f(M_{x_i}^*, M_{v_i}^*)$$

Therefore $(M_{x}^*, M_{v}^*)$ is the Nash equilibrium with mixed strategies, too.