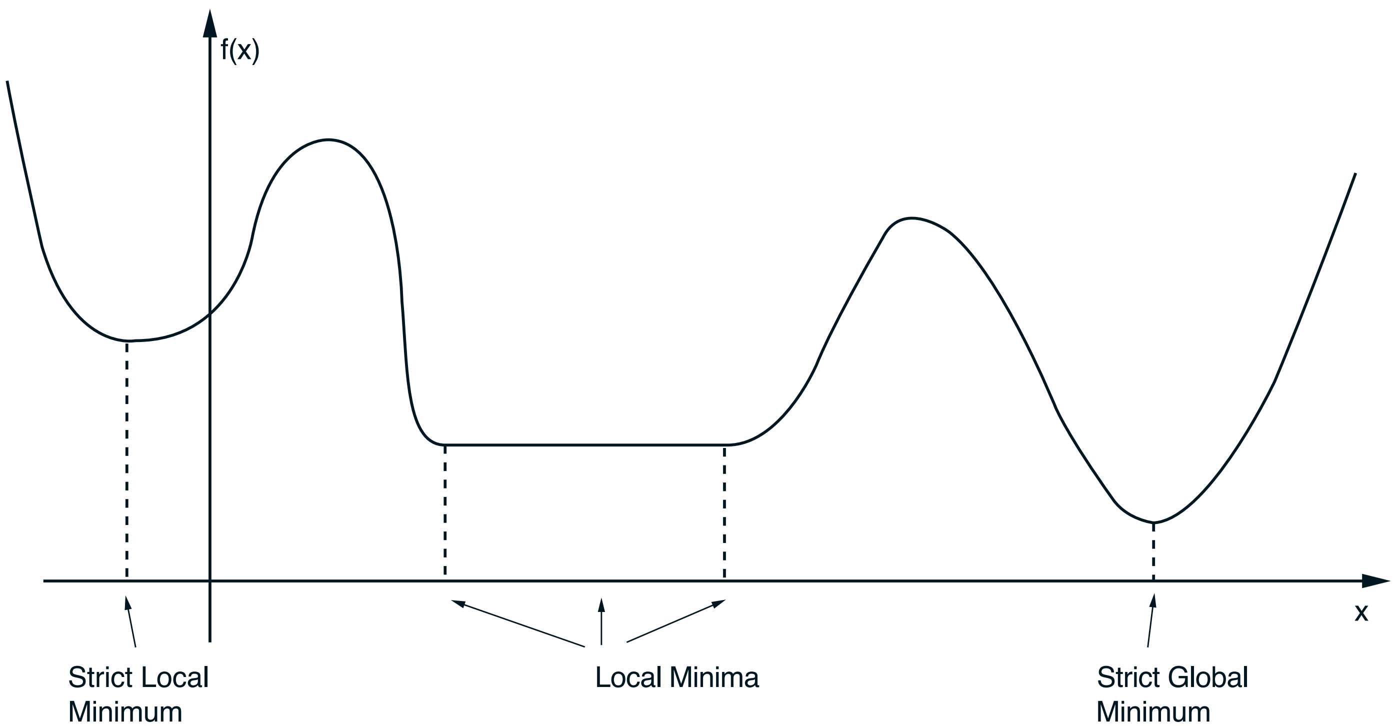
LOCAL AND GLOBAL MINIMA



Unconstrained local and global minima in one dimension.

NECESSARY CONDITIONS FOR A LOCAL MIN

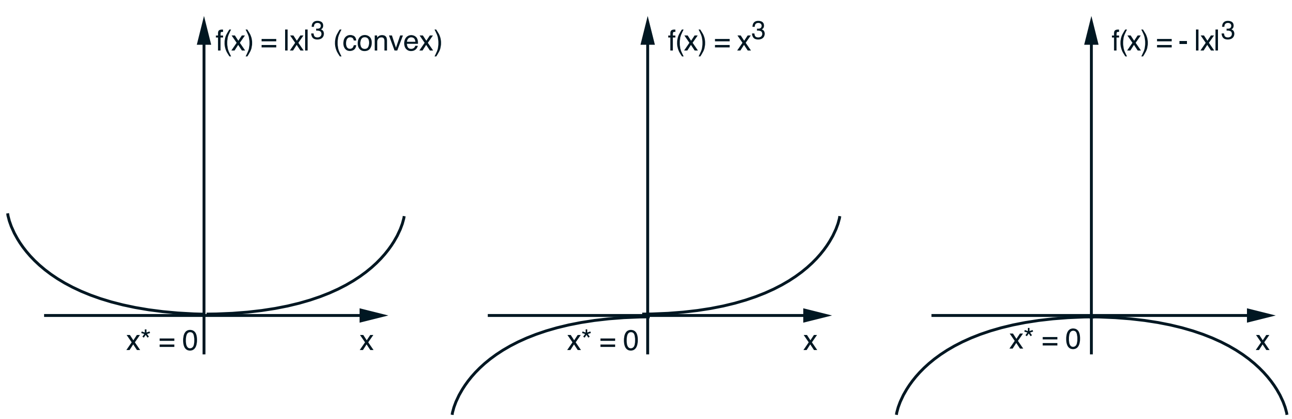
- **1st order condition:** Zero slope at a local minimum x^*

$$\nabla f(x^*) = 0$$

- **2nd order condition:** Nonnegative curvature at a local minimum x^*

$$\nabla^2 f(x^*) : \text{ Positive Semidefinite}$$

- There may exist points that satisfy the 1st and 2nd order conditions but are not local minima



First and second order necessary optimality conditions for functions of one variable.

PROOFS OF NECESSARY CONDITIONS

- **1st order condition** $\nabla f(x^*) = 0$. Fix $d \in \Re^n$. Then (since x^* is a local min), from 1st order Taylor

$$d'\nabla f(x^*) = \lim_{\alpha \downarrow 0} \frac{f(x^* + \alpha d) - f(x^*)}{\alpha} \geq 0,$$

Replace d with $-d$, to obtain

$$d'\nabla f(x^*) = 0, \quad \forall\, d \in \Re^n$$

- **2nd order condition** $\nabla^2 f(x^*) \geq 0$. From 2nd order Taylor

$$f(x^*+\alpha d) - f(x^*) = \alpha \nabla f(x^*)' d + \frac{\alpha^2}{2} d'\nabla^2 f(x^*) d + o(\alpha^2)$$

Since $\nabla f(x^*) = 0$ and x^* is local min, there is sufficiently small $\epsilon > 0$ such that for all $\alpha \in (0, \epsilon)$,

$$0 \leq \frac{f(x^* + \alpha d) - f(x^*)}{\alpha^2} = \tfrac{1}{2} d'\nabla^2 f(x^*) d + \frac{o(\alpha^2)}{\alpha^2}$$

Take the limit as $\alpha \to 0$.

SUFFICIENT CONDITIONS FOR A LOCAL MIN

- **1st order condition:** Zero slope

$$\nabla f(x^*) = 0$$

- **1st order condition:** Positive curvature

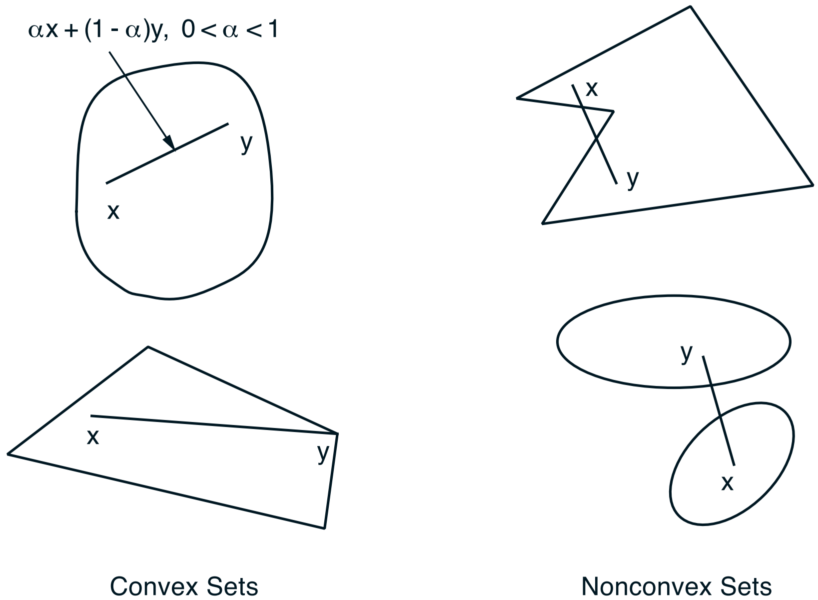
$$\nabla^2 f(x^*) : \text{Positive Definite}$$

- **Proof:** Let $\lambda > 0$ be the smallest eigenvalue of $\nabla^2 f(x^*)$. Using a second order Taylor expansion, we have for all d

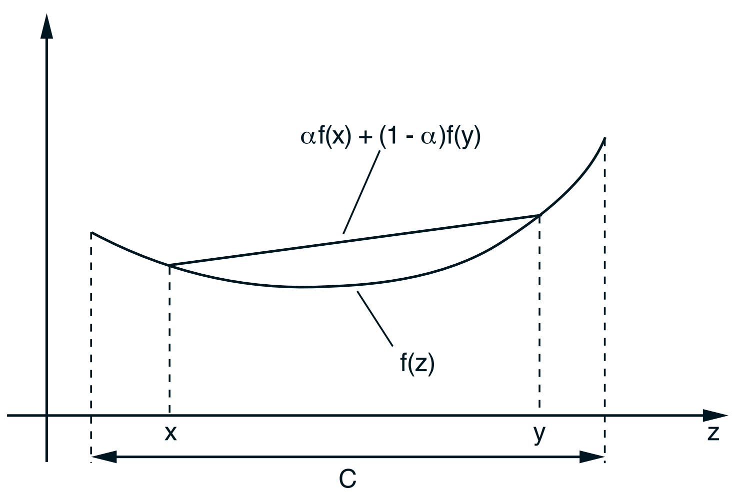
$$\begin{aligned} f(x^* + d) - f(x^*) &= \nabla f(x^*)'d + \frac{1}{2} d' \nabla^2 f(x^*) d \\ &\quad + o(\|d\|^2) \\ &\geq \frac{\lambda}{2} \|d\|^2 + o(\|d\|^2) \\ &= \left(\frac{\lambda}{2} + \frac{o(\|d\|^2)}{\|d\|^2} \right) \|d\|^2. \end{aligned}$$

For $\|d\|$ small enough, $o(\|d\|^2)/\|d\|^2$ is negligible relative to $\lambda/2$.

CONVEXITY



Convex and nonconvex sets.



A convex function. Linear interpolation underestimates the function.

MINIMA AND CONVEXITY

- Local minima are also global under convexity

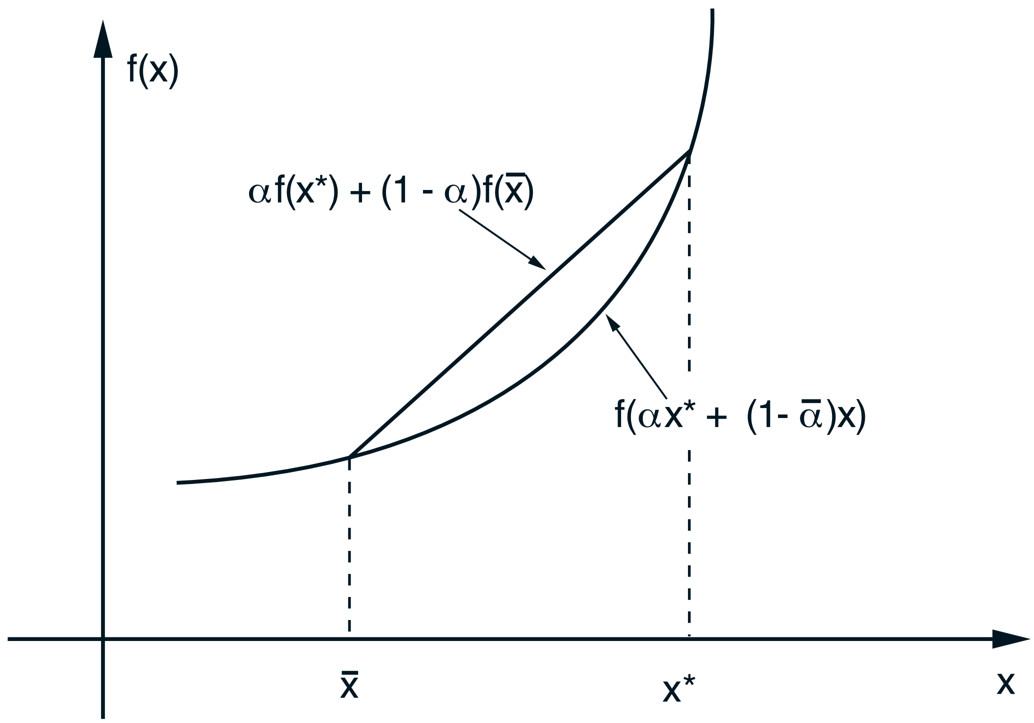


Illustration of why local minima of convex functions are also global. Suppose that f is convex and that x^* is a local minimum of f. Let $\overline{x}$ be such that $f(\overline{x}) < f(x^*)$. By convexity, for all $\alpha \in (0,1)$,

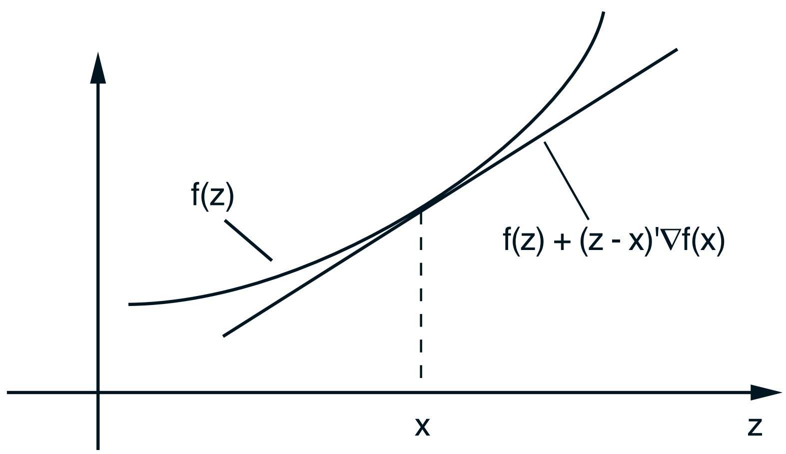
$$f\Big(\alpha x^* + (1-\alpha)\overline{x}\Big) \le \alpha f(x^*) + (1-\alpha) f(\overline{x}) < f(x^*).$$

Thus, f takes values strictly lower than $f(x^*)$ on the line segment connecting x^* with $\overline{x}$, and x^* cannot be a local minimum which is not global.

OTHER PROPERTIES OF CONVEX FUNCTIONS

- f is convex if and only if the linear approximation at a point x based on the gradient, underestimates f:

$$f(z) \geq f(x) + \nabla f(x)'(z - x), \qquad \forall\, z \in \Re^n$$



  – Implication:

$$\nabla f(x^*) = 0 \quad \Rightarrow x^* \text{ is a global minimum}$$

- f is convex if and only if $\nabla^2 f(x)$ is positive semidefinite for all x