If $\nabla f(x) \neq 0$, there is an interval $(0, \delta)$ of stepsizes such that

$$f\left(x - \alpha \nabla f(x) \right) < f(x)$$

for all $\alpha \in (0, \delta)$.

If $d$ makes an angle with $\nabla f(x)$ that is greater than 90 degrees,

$$\nabla f(x)'d < 0,$$

there is an interval $(0, \delta)$ of stepsizes such that $f(x + \alpha d) < f(x)$ for all $\alpha \in (0, \delta)$. 
PRINCIPAL GRADIENT METHODS

\[ x^{k+1} = x^k + \alpha^k d^k, \quad k = 0, 1, \ldots \]

where, if \( \nabla f(x^k) \neq 0 \), the direction \( d^k \) satisfies

\[ \nabla f(x^k)' d^k < 0, \]

and \( \alpha^k \) is a positive stepsize. Principal example:

\[ x^{k+1} = x^k - \alpha^k D^k \nabla f(x^k), \]

where \( D^k \) is a positive definite symmetric matrix

- Simplest method: Steepest descent
  \[ x^{k+1} = x^k - \alpha^k \nabla f(x^k), \quad k = 0, 1, \ldots \]

- Most sophisticated method: Newton’s method
  \[ x^{k+1} = x^k - \alpha^k (\nabla^2 f(x^k))^{-1} \nabla f(x^k), \quad k = 0, 1, \ldots \]
STEEPEST DESCENT AND NEWTON’S METHOD

Slow convergence of steepest descent

Fast convergence of Newton’s method w/ $\alpha^k = 1$.

Given $x^k$, the method obtains $x^{k+1}$ as the minimum of a quadratic approximation of $f$ based on a second order Taylor expansion around $x^k$. 
OTHER CHOICES OF DIRECTION

- Diagonally Scaled Steepest Descent

\[ D^k = \text{Diagonal approximation to } (\nabla^2 f(x^k))^{-1} \]

- Modified Newton’s Method

\[ D^k = (\nabla^2 f(x^0))^{-1}, \quad k = 0, 1, \ldots, \]

- Discretized Newton’s Method

\[ D^k = (H(x^0))^{-1}, \quad k = 0, 1, \ldots, \]

where \( H(x^k) \) is a finite-difference based approximation of \( \nabla^2 f(x^k) \)

- Gauss-Newton method for least squares problems: \( \min_{x \in \mathbb{R}^n} \frac{1}{2} \| g(x) \|^2 \). Here

\[ D^k = (\nabla g(x^k) \nabla g(x^k)')^{-1}, \quad k = 0, 1, \ldots \]