1 Elements of combinatorics

Consider the task of placing \( k \) balls in \( n \) cells, where at most one ball is allowed in a cell. The number of possible dispositions without repetitions (sometimes called permutations without repetitions) is

\[
D^k_n = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!},
\]

where \( 0! = 1 \) by definition. Equivalently, one can think of the ordered drawing of \( k \) balls from \( n \) numbered balls without replacement. Order matters.

**Example 1** The number of different three letter words one can form from \( \{A, B, C, D, E\} \) without repeating letters is \( 5 \cdot 4 \cdot 3 = 60 \).

**Example 2** We have a group on \( N \) persons. Assume that the probability of being born in any of the 365 days of the year is the same (we disregard leap years). Find the smallest \( N \) such that the probability that at least two individuals in the group have the same birthday is greater than \( 1/2 \).

It is easier to compute the probability \( p \) that all \( N \) persons have different birthdays. We get

\[
p = \frac{D^N_{365}}{D^N_{365}} = \frac{365 \cdot 364 \cdot 363 \cdots (365 - N + 1)}{365^N} = \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{N - 1}{365}\right).
\]
For $N$ small, we can ignore all the cross products, getting
\[ p \approx 1 - \frac{1 + 2 + \cdots + N - 1}{365} = 1 - \frac{N(N - 1)}{2 \cdot 365} = 1 - \frac{N(N - 1)}{730}. \]
For $N = 10$, we get $p \approx 0.877$, the correct value being 0.883. For larger $N$, we get a better approximation using logarithms and the fact that, for small positive $x$, we have \( \log(1 - x) \approx -x \). From (1), we then have
\[ \log p \approx -\frac{1 + 2 + \cdots + (N - 1)}{365} = -\frac{N(N - 1)}{730}. \]
For $N = 30$, this approximation yields 0.3037, whereas the correct value is $p = 0.294$. For $N = 22$ the correct value is 0.524 very close to $1/2$. In contrast to our intuition that leads us to choose large $N$'s, the answer to the problem is $N = 23$.

Consider now the different ways one can order $n$ objects: These are called permutations. It is a particular case of dispositions without repetitions when $k = n$. Indeed,
\[ P_n = D_n = \frac{n!}{0!} = n!. \]
Consider placing $k$ indistinguishable balls in $n$ cells, with at most one ball in each cell. Equivalently, consider drawing simultaneously $k$ balls from $n$ numbered. Here order does not matter. These are called combinations without repetitions. Their number is denoted by $C_k^n$ (or $C(n,k)$). Since $k$ objects may be ordered in $k!$ different ways, we have
\[ C_k^n \cdot k! = D_k^n = \frac{n!}{(n-k)!} \Rightarrow C_k^n = \frac{n!}{k!(n-k)!} := \binom{n}{k}. \]
For $k < 0$ or $k > n$, set \( \binom{n}{k} = 0 \).

**Example 3** Among the $2^n$ sequences of outcomes in $n$ coin tosses, there are \( \binom{n}{k} \) that have exactly $k$ heads. For instance, in five coin tosses, there are
\[ \binom{5}{2} = 10 \]
different ways to have exactly two heads.
2 Fibonacci's numbers

The binomial coefficient \( \binom{n}{k} \) (read “n choose k”) owns its name to the expansion

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.
\] (2)

Notice that the binomial theorem (2) implies, taking \( a = b = 1 \),

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\] (3)

This has an obvious interpretation in terms of \( n \) coin tosses in view of Example 3. Directly from the definition we get

\[
\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{n} = \binom{n}{0} = 1.
\]

It is not difficult to establish Pascal’s rule

\[
\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}, \quad 0 \leq k < n.
\] (4)

It follows from Pascal’s identity that the binomial coefficient is always a natural number.

**Exercise 1** Show that for \( n \) even \( \binom{n}{k} \) is maximum for \( k = n/2 \) and for \( n \) odd it is maximum for \( k = (n-1)/2 \) and \( (n+1)/2 \).

From Pascal’s identity (4), we get the Tartaglia’s Triangle (also named after Pascal, Yang Hui, Omar Khayyám, Pingala, Stiefel,...), see Table 1. The triangle’s rows are precisely the coefficients which arise in the binomial expansion (2). Here are some of the triangle properties.

1. The triangle is symmetric;
2. by (3), the sum of the \( n^{th} \) row is \( 2^n \);
3. the digits in the \( n^{th} \) row compose the number \( 11^n \). For \( n = 5 \), this amounts to \( 11^5 = 1 \cdot 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 10 \cdot 10^2 + 5 \cdot 10 + 1 \cdot 10^0 \). Similarly for \( n > 5 \);
Table 1: Rows 0 – 4 in Tartaglia’s Triangle.

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4. sums over “diagonals” yield the Fibonacci numbers Figure 1, i.e.

\[
\sum_{k=0}^{n} \binom{n-k}{k} = F(n+1),
\]

where the Fibonacci (Leonardo Pisano was posthumously named filius Bonacci) numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … are defined by the recursion

\[
F(n+1) = F(n) + F(n-1), \quad F(0) = 0, \quad F(1) = 1.
\]

As observed by Kepler,

\[
\lim_{n \to \infty} \frac{F(n+1)}{F(n)} = \varphi,
\]

where \( \varphi = \frac{1+\sqrt{5}}{2} \approx 1.618 \), called golden ratio (sectio aurea) is the only positive solution of the characteristic equation \( x^2 - x - 1 = 0 \) (\( x > 0 \) is such that \( 1 : x = x : (1 + x) \)). This ratio, has fascinated scientists, artists, musicians, etc. since antiquity as it appears in mathematics, architecture, aesthetics, and so on.\(^1\)

\(^1\)This sequence, described earlier in Indian Mathematics, was introduced to western science by Leonardo Pisano in 1202 in his Liber Abaci, see Figure 2.

\(^2\)One of the pioneers of probability was Luca Pacioli who taught mathematics to Leonardo da Vinci. He wrote De Divina Proportione, published in 1509, which is mostly devoted to \( \varphi \) (Leonardo did the engraving). Pacioli also gave seminal contributions to the field now known as accounting and wrote a chess treatise De Ludo Scacchorum around year 1500 which was only discovered in an aristocratic private library in 2006! The drawings may be due to Leonardo, see Figure 3.
3  Fibonacci’s equation

Fibonacci’s equation is the simple recursion

\[ y(n) - y(n-1) - y(n-2) = 0 \]  \hspace{1cm} (7)

supplemented with the initial conditions

\[ y(0) = 0, \quad y(1) = 1. \]  \hspace{1cm} (8)

Since every number is the sum of the two previous ones, if we give as initial conditions two not both zero natural numbers, it follows that the solution is an increasing sequence of natural numbers. If we look at the general solution of (7), however, this might appear somewhat surprising. Indeed, since the two solutions of the characteristic equation \( x^2 - x - 1 = 0 \) are \( \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \), the general solution is

\[ y(n) = c_1 \lambda_1^n + c_2 \lambda_2^n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n. \]  \hspace{1cm} (9)

Imposing the initial conditions (8), we readily get

\[ c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = -\frac{1}{\sqrt{5}}. \]

Thus, the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... may be (somewhat surprisingly) obtained as

\[ y(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]. \]  \hspace{1cm} (10)

This formula is due to Abraham De Moivre. If we employ the binomial theorem (2). We get

\[
y(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] = \frac{1}{2^n \sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k - \frac{1}{2^n \sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k
\]

\[
= \frac{1}{2^{n-1}} \sum_{k=1, k \text{ odd}}^{n} \binom{n}{k} (\sqrt{5})^k [1 - (-1)^k]
\]

\[
= \frac{1}{2^{n-1}} \sum_{k=1, k \text{ odd}}^{n} \binom{n}{k} (5)^{k-1/2}.
\]  \hspace{1cm} (11)
Figure 1: “The flight of numbers” by Mario Merz on the Mole Antonelliana, Turin, representing Fibonacci’s sequence
Figure 2: A page of Fibonacci’s Liber Abaci from the Biblioteca Nazionale di Firenze showing (in box on right) the Fibonacci sequence with the position in the sequence labeled in Latin and Roman numerals and the value in Hindu-Arabic numerals.
Figure 3: A page of *De ludo scacchorum* by Pacioli with a drawing that some attribute to Leonardo.