Consider a process that produces items. Suppose that the process is in control, and the items produced have a normal distribution with mean \( \mu \) and standard deviation \( \sigma \), and that each item is measured. Suppose that the items produced have a normal distribution with mean \( \mu \) and standard deviation \( \sigma \), and that each item is measured.

**Example 8.4 Qualify Control**

**Simulation estimation**

Here the problem is similar, even in quite simple models of using the new silver.

However, before proceeding these variance reduction techniques, let us discuss.

In this chapter we present a variety of different methods that can be used to estimate variance in the case of a simulator. These methods allow us to obtain an improved estimate of the variance.

Hence, if we can obtain a different unbiased estimate of \( \sigma^2 \), we would obtain an improved estimate of the variance.

\[
\frac{\hat{\sigma}^2}{(X)^{\hat{\sigma}^2}} = \frac{\hat{\sigma}^2}{(X)^{\hat{\sigma}^2}} = \frac{\hat{\sigma}^2}{(X)^{\hat{\sigma}^2}} = \frac{\hat{\sigma}^2}{(X)^{\hat{\sigma}^2}}
\]

Variance. That is...

In a typical scenario for a simulation study, one is interested in determining...
8.1 The Use of Arithmetic Variables

Random variables—A formalized view.

In a simulation, the number of random variables used depends on the number of random variables needed. The number of random variables needed is the number of random variables used in the simulation. The number of random variables used is the number of random variables used in the simulation.

Since the number of random variables in a simulation is the number of random variables used in the simulation, the number of random variables used in a simulation is the number of random variables used in the simulation. The number of random variables used in a simulation is the number of random variables used in the simulation.

In addition, because each random variable is used only once, the number of random variables used in a simulation is the number of random variables used in the simulation. The number of random variables used in a simulation is the number of random variables used in the simulation.

Now, if we can simulate the above claim by simulation, we can say that our claim is correct.

The process is executed on a computer that can simulate the claim. The computer is a machine that can simulate the claim. The computer is a machine that can simulate the claim.

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The process is executed on a computer that can simulate the claim. The computer is a machine that can simulate the claim. The computer is a machine that can simulate the claim.
The bridge survives. Hence the parallel system works if at least one of its components works.

\[
\begin{align*}
\prod_{i}^{n} \phi_i &= (\phi_1 \cdot \ldots \cdot \phi_n) \\
\prod_{i}^{n} \bar{\phi}_i &= \overline{\prod_{i}^{n} \phi_i}
\end{align*}
\]

\[
\prod_{i}^{n} \phi_i \prod_{i}^{n} \bar{\phi}_i = (\phi_1 \cdot \ldots \cdot \phi_n) \overline{\prod_{i}^{n} \phi_i} = 0
\]

The function \((\phi_1 \cdot \ldots \cdot \phi_n) \overline{\prod_{i}^{n} \phi_i} = 0\)

Example 88: Simulating the Redundancy Function Consider a system of \(n\) components, each of which is either function or faulty. Let \(u_i\) be the survival of component \(i\) and denote the system survival function by \(\delta(x)\).

\[
\delta(x) = \sum_{i=1}^{n} u_i^{x_i} \prod_{i=1}^{n} \bar{u}_i^{1-x_i}
\]

The function \(\delta(x)\) is the probability that the system survives when each component survives with probability \(u_i\).
The use of arithmetic variances

Example of simulating a queueing system

When we are interested in simulating a queueing system, we first decide on the type of the queueing components. Consider a queueing system of two types of components, each type consisting of n servers. The service time for each type of component is exponentially distributed with parameters \( \lambda_1 \) and \( \lambda_2 \), respectively.

The service discipline is first-in, first-out (FIFO). The system operates in a continuous-time Markov process with state space \( S = \{0, 1, 2, \ldots\} \), where 0 represents an empty system and \( n \) represents a system with \( n \) customers in service.

The transition rates are given by

\[
\begin{align*}
\lambda_{01} &= \lambda_1 + \lambda_2 \\
\lambda_{10} &= \lambda_1 \\
\lambda_{20} &= \lambda_2 \\
\lambda_{i0} &= \sum_{j=i+1}^{\infty} \lambda_j \\
\lambda_{ij} &= \lambda_{ji} = \lambda_i \\
\lambda_{ii+1} &= 0
\end{align*}
\]

The steady-state distribution of the process is given by

\[
\pi = \begin{pmatrix}
\pi_0 & \pi_1 & \pi_2 & \cdots
\end{pmatrix}
\]

where \( \pi_i \) is the steady-state probability of having \( i \) customers in service.

To simulate the system, we need to generate random numbers to simulate the service times and the interarrival times. We can use the inverse transform method to generate these random numbers.

1. Generate a random number \( u \) from a uniform distribution on \( (0, 1) \).
2. Use the service time distribution to generate the service time \( T \).
3. Use the interarrival time distribution to generate the interarrival time \( I \).

For a parallel system, the equations are similar, but the service rates and the interarrival rates are different.

\[
\begin{align*}
\lambda_{01} &= \lambda_1 + \lambda_2 \\
\lambda_{10} &= \lambda_1 \\
\lambda_{20} &= \lambda_2 \\
\lambda_{i0} &= \sum_{j=i+1}^{\infty} \lambda_j \\
\lambda_{ij} &= \lambda_{ji} = \lambda_i \\
\lambda_{ii+1} &= 0
\end{align*}
\]

For a series system, the equations are similar, but the service rates and the interarrival rates are different.

\[
\begin{align*}
\lambda_{01} &= \lambda_1 + \lambda_2 \\
\lambda_{10} &= \lambda_1 \\
\lambda_{20} &= \lambda_2 \\
\lambda_{i0} &= \sum_{j=i+1}^{\infty} \lambda_j \\
\lambda_{ij} &= \lambda_{ji} = \lambda_i \\
\lambda_{ii+1} &= 0
\end{align*}
\]

For a queueing system, we can write

The equations for the queueing system can be written as

\[
\begin{align*}
\pi_0 &= \sum_{i=1}^{\infty} \pi_i \\
\pi_i &= \pi_{i-1} \lambda_i / \lambda_1 + \pi_{i+1} \lambda_i / \lambda_2 \\
\pi_1 &= \pi_0 \lambda_1 / \lambda_1 + \pi_2 \lambda_1 / \lambda_2 \\
\pi_2 &= \pi_1 \lambda_1 / \lambda_2 + \pi_3 \lambda_1 / \lambda_2 \\
&\vdots
\end{align*}
\]

To solve these equations, we can use the matrix method.

\[
\begin{pmatrix}
\pi_0 \\
\pi_1 \\
\pi_2 \\
\vdots
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 & \lambda_2 & 0 & \cdots \\
\lambda_1 & \lambda_1 & \lambda_2 & \cdots \\
\lambda_1 & \lambda_2 & \lambda_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\pi_0 \\
\pi_1 \\
\pi_2 \\
\vdots
\end{pmatrix}
\]

The solution to this system of equations gives the steady-state probabilities \( \pi_i \) for each state of the queueing system.
Example 8.1. The Use of Mathematical Notations

Example 8.2. Variance Reduction Techniques

\[
(1-\eta) < \eta, \quad \eta = u : u \text{ min} = N
\]

Let be the first one that is greater than its immediate predecessor. Then it's

\[
6900.9 = \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2}
\]

whereas the use of the arithmetic variances and of the variance of

\[
0.2405 = \int_0^1 \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2}
\]

we see that the use of independent random numbers results in a variance of

\[
2.420 = \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2}
\]

Also, because

\[
\frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2}
\]

begin not to be a variance reduction, whose value we now determine. To

\[
\frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2}
\]

Suppose we were interested in using simulation to estimate

\[
\frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2} \cdot \frac{z}{(n-1-\eta)^2}
\]

be governed by the use of arithmetic variances.

The following example illustrates the sort of improvement that can sometimes
The Use of Control Variates

8.2 Variance Reduction Techniques

The use of control variates is a second set of random variables that are correlated with the random variables used to compute the expected value of the function of interest. By using these control variates, we can reduce the variance of our estimator, leading to a more accurate estimate of the expected value.

In the case of a normal random variable, let $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. We define a control variate $Y$ as a random variable that is correlated with $X$ and has mean $\mu_Y$. The estimator for the expected value of $X$ can then be written as:

$$ \hat{E}[X] = \hat{E}[X + cY] - c \hat{E}[Y] $$

where $c$ is a constant that minimizes the variance of the estimator.

For a normal distribution, the variance of the estimator is minimized when $c = \frac{\sigma_X^2}{\sigma_Y^2}$.

$$ \frac{\sigma_X^2}{\sigma_Y^2} = \frac{\hat{E}[X] - \hat{E}[Y]}{\hat{E}[X] - \hat{E}[Y]} $$

Simplifying this equation, we get:

$$ \frac{\sigma_X^2}{\sigma_Y^2} = \frac{\hat{E}[X]^2 - \hat{E}[Y]^2}{\hat{E}[X]^2 - \hat{E}[Y]^2} $$

Thus, the variance of the estimator is:

$$ \text{Var}[\hat{E}[X]] = \text{Var}[\hat{E}[X + cY] - c \hat{E}[Y]] = c^2 \sigma_Y^2 $$

where $c^2 \sigma_Y^2$ is the variance of the control variate $Y$.

Hence, the variance of the estimator can be reduced by using a control variate with a high correlation to the random variable of interest.

$$ \text{Var}[\hat{E}[X]] = \text{Var}[\hat{E}[X + cY] - c \hat{E}[Y]] = c^2 \sigma_Y^2 $$

Implying that:

$$ c = \frac{\sigma_X}{\sigma_Y} $$

Some algebra, which uses the previously obtained results for $\hat{E}[N]$, $\hat{E}[N+1]$, and $\text{Var}[N]$, we obtain that:

$$ \hat{E}[N] + \hat{E}[N+1] = \hat{E}[N] + \hat{E}[1] $$

Hence:

$$ \text{Var}[\hat{E}[X]] = \text{Var}[\hat{E}[X + cY] - c \hat{E}[Y]] = c^2 \sigma_Y^2 $$

Hence, the use of control variates reduces the variance of our estimator, leading to a more accurate estimate of the expected value.
\[ S \xrightarrow{\text{in}} = q \]

where the number of arrivals by time \( t \) is the number of arrivals by time \( t \) that are served.

Example 89

Consider a queuing system in which customers arrive in

\[ d \xrightarrow{\text{in}} = \left[ S \xrightarrow{\text{in}} \right] \exists \]

where \( S \rightarrow \exists \) is the service time of the \( i \)th customer.

The service times are independent and identically distributed (i.i.d.).

Lemma 9

A simple linear regression model can be written as:

\[ y = \beta_0 + \beta_1 x + \epsilon \]

The variance of the linear model is:

\[ \sigma^2 = \frac{\sum (y_i - \bar{y})^2}{n-2} \]

where \( \bar{y} \) is the sample mean.

The variance of the regression coefficients is:

\[ \text{Var}(\beta_1) = \frac{\sigma^2}{\sum x_i^2} \]

where \( x_i \) are the predictor variables.

Example 90

Suppose we are interested in estimating the relationship between two variables, \( x \) and \( y \). We have a sample of data points \( (x_i, y_i) \) for \( i = 1, \ldots, n \).

The correlation coefficient between \( x \) and \( y \) is:

\[ r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} \]

where \( \bar{x} \) and \( \bar{y} \) are the sample means.

The linear regression model can be written as:

\[ y = \beta_0 + \beta_1 x + \epsilon \]

where \( \beta_0 \) and \( \beta_1 \) are the regression coefficients.

The variance of the regression coefficient is:

\[ \text{Var}(\beta_1) = \frac{\sigma^2}{\sum x_i^2} \]

where \( \sigma^2 \) is the variance of the error term.

The confidence interval for \( \beta_1 \) is:

\[ \beta_1 \pm t_{\alpha/2, n-2} \frac{\sigma}{\sqrt{\sum x_i^2}} \]

where \( t_{\alpha/2, n-2} \) is the critical value from the \( t \)-distribution with \( n-2 \) degrees of freedom.
Example 8.1: Using a block diagonal first order estimation of $[X]$, the equation of a block diagonal is overplotted with the

$\text{Example 8.1: Using a block diagonal first order estimation of } [X], \text{ the equation of a block diagonal is overplotted with the}$

$\left(\begin{array}{c}
\sigma^2 \\
\sigma^2 \\
\end{array}
\right) X + X$

$\text{Example 8.1: Using a block diagonal first order estimation of } [X], \text{ the equation of a block diagonal is overplotted with the}$

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\sigma^2 \\
\sigma^2 \\
\end{array}
\right) X + X$

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\sigma^2 \\
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\sigma^2 \\
\sigma^2 \\
\end{array}
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\sigma^2 \\
\sigma^2 \\
\end{array}
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$\left(\begin{array}{c}
\sigma^2 \\
\sigma^2 \\
\end{array}
\right) X + X$

$\text{Example 8.1: Using a block diagonal first order estimation of } [X], \text{ the equation of a block diagonal is overplotted with the}$

$\left(\begin{array}{c}
\sigma^2 \\
\sigma^2 \\
\end{array}
\right) X + X$
9.3 \frac{(M \cdot X)^{\omega} \cdot C - (M)^{\omega} \cdot A + (X)^{\omega} \cdot A}{(M \cdot X)^{\omega} \cdot C - (M)^{\omega} \cdot A} = \sigma

\text{variance is given by formula } \sigma^2 = \frac{\text{variance of error}}{\text{variance of total}}\cdot \text{where}

\text{the best such estimator which is obtained by minimizing the}

M(\sigma - 1) + X \sigma

\text{Consider any unbiased estimator of the form}

\theta = [M] \cdot X = [X] \cdot \theta

\text{then we may define}

M \cdot X

\text{and suppose that the variance of } \theta \text{ is the variance of the control variable. In this case, the}

\text{variance of } \theta \text{ is equal to the variance of the control variable.}

\text{The problem of finding the control variable that is the}

\text{variance of the variance of the control variable is then equivalent to}

\text{minimizing the variance of the control variable.}

\text{This estimate can be obtained from a simple multiple regression program.}

\text{The estimate is given by}

\text{where}

\text{is the least squares estimator of } \theta \text{ and is the dependent variable.}

\text{and}

\text{is a control variable. Of course,}

0 = \left[ (Z - M)^{\omega} \cdot A \right] \cdot \theta

\text{takes the form}

\text{and is not difficult to show that}

\text{and}

\text{and}

\text{and}

\text{is the expected number of control cards dealt at least 19}

\text{and}

\text{is the expected number of control cards dealt and at least 19}

\text{in the example above. Since the expected number of control cards dealt and at least 19 is large, the probability of getting more than 19 cards is small.}

\text{Let } \gamma \text{ denote the number of hands played in a game, and let } \gamma \text{ denote the}

\text{number of successful outcomes. Then, if } \gamma \text{ is the number of successful outcomes, we can use a control variable that is}

\text{the number of successful outcomes.}

\text{If we assume that the player is employing some fixed strategy which}

\text{may be calculated using a computer program for the multiple regression model.}

\text{When multiple control variables are used, the computation can be performed}
We now illustrate the use of conditioning by a series of examples.

To see this, we set no additional variance reduction is possible by conditioning the estimator $X$ and $E[X]$. From (4.8)

$$
E[(X-E)(X-E)] = \Sigma X
$$

This follows from the unbiased estimator $\hat{E}[X]$. Since $E[X]$ is a constant, we have

$$
E[(X-E)(X-E)] = (E[X]-E)^2 = (E[X]-E)^2 = C^2\Sigma X
$$

On the other hand,

$$
E[(X-E)(X-E)] = (E[X]-E)^2
$$

Thus, we see that no additional variance reduction is possible by conditioning the estimator $X$ and $E[X]$.

8.3 Variance Reduction by Conditioning

We now show that if $X = 0$, then

$$(\hat{X} - X)^2 = (\hat{X} - X)^2 + (X - E)^2$$

This follows from

$$
E[(X-E)(X-E)] = (E[X]-E)^2 = (E[X]-E)^2
$$

Of course, the above equation is true because $X = 0$. However, for a general estimator $\hat{X}$, the best estimator of the form $\hat{X} = \hat{X} - X + X$ is

$$(\hat{X} - X)^2 = (\hat{X} - X)^2 + (X - E)^2$$

This follows from

$$
E[(X-E)(X-E)] = (E[X]-E)^2 = (E[X]-E)^2
$$

Remarks

To understand why the conditional expectation expression is supposed to be the best estimator of the form $\hat{X} = \hat{X} - X + X$, we can proceed as follows: First, we observe the unbiased nature of the estimator $\hat{X}$. Next, we note that we are optimizing the conditional expectation of $\hat{X}$ with respect to $X$. To do this, we need to find the best estimator $\hat{X}$ of the form $\hat{X} = \hat{X} - X + X$, where $X = 0$. Since $\hat{X} = \hat{X} - X + X$, the best estimator is $\hat{X}$.

8.3 Variance Reduction by Conditioning

Now if $E[X] = 0$, then $\hat{X} = \hat{X}$.

The unbiased estimator can be written as

$$
\hat{X} = (\hat{X} - X) + X
$$

where $\hat{X}$ is the unbiased estimator of $E[X]$. Since $\hat{X} = \hat{X}$, we have

$$
(\hat{X} - X) + X = (\hat{X} - X) + X
$$

Therefore, the unbiased estimator of $E[X]$ is

$$
\hat{X} = (\hat{X} - X) + X
$$

To see the way in which the unbiased estimator between two unbiased estimators $\hat{X}$ and $\hat{X}$ is

$$
(\hat{X} - \hat{X}) + X = (\hat{X} - \hat{X}) + X
$$

Now if $E[X] = 0$, then $\hat{X} = \hat{X}$.

8.3 Variance Reduction by Conditioning
\[ I \geq X \implies \{ \text{if } I \geq X \text{ then do } \} = I \]

Then, the upper limit of a random variable \( X \) is a moment with mean \( \mu \) and variance \( \sigma^2 \).

How can we use simulation to efficiently estimate the mean and variance of \( X \)?

**Example 8.8**

**Imprecise Implication:** (Example 12.5)

We use simulation to estimate the covariance between two random variables. The covariance can be estimated by using the simulation to estimate the covariance formula:

\[
\text{Cov}(X, Y) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})
\]

where \( \bar{X} \) and \( \bar{Y} \) are the sample means.

Hence, we can estimate the correlation coefficient as:

\[
\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\text{Var}(X) \cdot \text{Var}(Y)}
\]

**Example 8.9**

**Imprecise Implication:** (Example 12.6)

We use simulation to estimate the probability of an event. The probability can be estimated by using the simulation to estimate the proportion of times the event occurs:

\[
P(A) = \frac{1}{n} \sum_{i=1}^{n} I(A)
\]

where \( I(A) \) is an indicator function that is 1 if the event occurs and 0 otherwise.

Hence, we can estimate the probability of a complex event as:

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

**Example 8.10**

**Imprecise Implication:** (Example 12.7)

We use simulation to estimate the integral of a function. The integral can be estimated by using the simulation to estimate the average value of the function:

\[
\int_{a}^{b} f(x) \, dx \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i)
\]

where \( x_i \) are random samples from the interval \([a, b]\).
\[ \{m \geq N\} \cap \{w \geq N\} \neq \emptyset \]

\[ \{w \leq N\} \cap \{w \leq N\} \neq \emptyset \]

Now,

\[ \{w \leq N\} \cap \{w \leq N\} = \{w \leq N\} \]

Hence,

\[ W \leq N \iff 1 = 1 \]

In other words, the random variable \( W \) follows a Poisson distribution, where the parameter \( \lambda \) is the expected number of events within a given interval. The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event.

**Example:** Let \( X \) be a sequence of independent and identically distributed Poisson random variables with parameter \( \lambda \). The sum of these random variables, \( Y = X_1 + X_2 + \cdots + X_n \), is also a Poisson random variable with parameter \( n\lambda \). This property allows us to combine random variables into a single Poisson distribution when they are independent and identically distributed.

The variance of a Poisson random variable is equal to its mean. Therefore, if \( X \) is a Poisson random variable with mean \( \lambda \), then its variance is also \( \lambda \). This is a key property of the Poisson distribution, as it links the mean and variance in a simple manner. The Poisson distribution is often used to model the number of events occurring in a fixed interval of time or space when the events are rare and independent.

The variance-reduction techniques discussed in this chapter are particularly useful for simulating Poisson processes, where the number of events within a given interval follows a Poisson distribution. These techniques can help reduce the variance of the simulation results, leading to more accurate estimates of the expected outcomes. By applying these methods, we can improve the efficiency and accuracy of simulations involving Poisson processes.
Example 80

The value of \( \beta \) is given by Equation (8-9) and can be estimated from:

\[
(\mu - s_p(\lambda)) \int_{\lambda}^{\infty} (1 + s_p(s)) s_p(s) s^\lambda d s = 9
\]

Therefore, \( s_p(s) = \int_{\lambda}^{\infty} (1 + s_p(s)) s^\lambda d s \)

“The expected number of occurrences in a given time period is the expected number of occurrences in the system.

Example in a Finite Capacity Queueing Model

Consider a finite capacity queueing model where the number of customers in the system is limited and the customers arrive according to a Poisson process. The system can be modeled using a Markov chain. The state of the system at time \( t \) is determined by the number of customers in the system, and the transition rates are given by the Poisson distribution. The steady-state probabilities can be found by solving the balance equations for the Markov chain. The expected number of customers in the system is given by the sum of the product of the steady-state probabilities and the number of customers in each state. The expected waiting time in the system can also be calculated using the Little's Law.

\[
\text{Expected number of customers in the system} = \sum_{n=0}^{\infty} n P_n
\]

\[
\text{Expected waiting time in the system} = \sum_{n=1}^{\infty} n P_n
\]

By applying these formulas, we can determine the performance measures of the queueing system, such as the expected number of customers in the system, the expected waiting time, and the expected waiting time in the queue.
Consider a set of \( n \) cells, with cell \( I \) having width \( \ell \). The expected number of cells that are not accidentally deleted before a set of \( \ell \) rows is not below, where the equality used the independence of the \( \ell \) rows. We obtain

\[
\sum_{\ell \geq N} \left( 1 - \frac{n}{\ell} \right) \leq \sum_{\ell > \left( \frac{\ln \rho}{n} \right)} \left( 1 - \frac{n}{\ell} \right).
\]

Therefore

\[
\{ \ell \geq N \} \quad \text{if and only if} \quad \left\{ \ell > \left( \frac{\ln \rho}{n} \right) \right\}
\]

This is

\[
\{ \ell > \left( \frac{\ln \rho}{n} \right) \} \quad \text{if and only if} \quad \left\{ \ell > \left( \frac{\ln \rho}{n} \right) \right\}
\]

For \( \ell \geq N \), all the vacant cells have been filled by time \( t \).
is of the form \( I(t) \). Let the time from \( t \) until the next renewal be

\[
\eta = \frac{1}{\lambda} + \frac{\eta}{\lambda}
\]

Thus, for large \( \lambda \), the best control action estimated of the above type is close to

\[
\eta = \frac{[\eta - (1+i)\lambda]}{\lambda} = \frac{[\eta - (1+i)\lambda]}{\lambda} \approx \eta
\]

comparing bounded—so far I have

because these improve with \( \eta \) increases, the other terms will

Now for \( i \), let the control action estimated of the above type be

\[
\frac{[\eta - (1+i)\lambda]}{\lambda} \approx \eta
\]

The best is given by

\[
(1+i)\lambda = 1+\eta\lambda
\]

...and so the above control action estimated can be written

\[
(1+i)\lambda = 1+\eta\lambda
\]

Figure 8.2. x = c

Estimating the Expected Number of Renewals by Time \( T \)

Let the first renewal time be \( (0,1) \) and the second renewal \( (i+1) \). Let the renewal be

\[
T = \frac{1}{\lambda}
\]

Support that the \( \eta \) are occurring randomly in time. Let \( T \) denote the time of

There is a renewal time at \( 0 \) and the second at \( \eta \), and in general, the number of

---

8.3 Variance Reduction by Conditioning

---

1. Step 1: Let \( T \) be the time of the \( w^{th} \) renewal. Let \( T' \) be the time of the \( w + 1 \) renewal.
2. Step 2: Let \( w + 1 \) be the \( w + 1 \) renewal. Let \( T' \) be the time of the \( w + 1 \) renewal.
3. Step 3: Count the random numbers
Continued variance formula where $E[X] = \mu_v$

$$E[(X|X)\,m_A\,A] = (X|X)\,m_A$$

Because $m_A$ is the variance of $X$.

$$E[(X|X)\,m_A\,A] = (X|X)\,m_A$$

where $E[X] = \mu_v$

Continuously, we have the proceeding and thus $X \sim 1, \ldots, X \sim 1$, are independent.

$$\frac{d\mu}{\mu} = (X|X)\,m_A$$

The estimator $\tilde{\mu}$ is called a smoothed sampling estimator of $E[X]$.

$$\tilde{\mu} = \frac{\sum i \cdot x_i}{n}$$

Figure 8.3. An example of a smoothed sampling estimator where $E[X] = \mu_v$.

8.4 Smoothed Sampling

Equation (8.3)

$$\frac{1}{n^2} \sum (i^2 - \bar{x})^2$$

Thus, for large $n$, a better estimator of $E[X]$ is given by

$$[\tilde{\mu}]^2 = \frac{\sum (i^2 - \bar{x})^2}{n}$$

Hence, with $\tilde{\mu}$ defined as above to equal $\bar{x}$.

The above estimator can further be improved upon by the use of conditioning.
\[ \begin{align*}
\mathbb{E}_d & = \frac{1}{8} \mathbb{E}(X) + \frac{1}{8} \mathbb{E}(Y) \\
& = \frac{1}{8} (16) + \frac{1}{8} (8) = 3
\end{align*} \]

Consequently, the variance of \( X \) is:

\[ \text{Var}(X) = \mathbb{E}_d^2 - \mathbb{E}_d = 3^2 - 3 = 6 \]

We can use this to estimate the variance of \( Y \) as:

\[ \text{Var}(Y) = \mathbb{E}_d^2 - \mathbb{E}_d = 3^2 - 3 = 6 \]

Thus, the variance of \( X \) is:

\[ \text{Var}(X) = \frac{1}{8} \mathbb{E}(X) + \frac{1}{8} \mathbb{E}(Y) = \frac{1}{8} (16) + \frac{1}{8} (8) = 3 \]

The variance of \( Y \) is:

\[ \text{Var}(Y) = \frac{1}{8} \mathbb{E}(Y) + \frac{1}{8} \mathbb{E}(X) = \frac{1}{8} (8) + \frac{1}{8} (16) = 3 \]

Example 8.4.7

When a sample is obtained from a normal distribution, the sample mean is an unbiased estimator of the population mean. However, the sample variance is biased because the expected value of the sample variance is smaller than the population variance. To obtain an unbiased estimator of the population variance, we can use the following formula:

\[ \text{Var}(X) = \frac{1}{n-1} \sum (X_i - \bar{X})^2 \]

Remark

The variance of the sample mean is approximately equal to the variance of the population divided by the sample size. This is known as the law of large numbers.

\[ \text{Var}(\bar{X}) = \frac{\text{Var}(X)}{n} \]

The variance of the sample mean is:

\[ \text{Var}(\bar{X}) = \frac{1}{8} \frac{1}{8} = \frac{1}{64} \]

The variance of the sample variance is:

\[ \text{Var}(S^2) = \frac{(n-1)^2}{n(n-1)} \text{Var}(X) \]

Therefore, the variance of the sample variance is:

\[ \text{Var}(S^2) = \frac{(8-1)^2}{8(8-1)} \cdot \frac{1}{64} = \frac{7^2}{64} \]

The variance of the sample variance is:

\[ \text{Var}(S^2) = \frac{7^2}{64} \]

This shows that the sample variance is an unbiased estimator of the population variance.

\[ \mathbb{E}(S^2) = \sigma^2 \]

Therefore, the variance of the sample variance is:

\[ \text{Var}(S^2) = \frac{(8-1)^2}{8(8-1)} \cdot \frac{1}{64} = \frac{7^2}{64} \]

The variance of the sample variance is:

\[ \text{Var}(S^2) = \frac{7^2}{64} \]

This shows that the sample variance is an unbiased estimator of the population variance.
Suppose that we have unbiased independent random variables $X$ and $Y$.

where

$$
\left(\frac{u}{1-f+\lambda}\right)^{1/f} \lesssim \left(\frac{u}{1-f+\lambda}\right)^{1/f} \lesssim \frac{u}{1-f+\lambda}
$$

Hence, we can estimate $\lambda$ using the unbiased estimator

$$
\left(\frac{u}{1-f+\lambda}\right)^{1/f} \lesssim \frac{u}{1-f+\lambda}
$$

Example 8. In Example 8, we showed that

$$
\left(\frac{u}{1-f+\lambda}\right)^{1/f} \lesssim \frac{u}{1-f+\lambda}
$$

is a better estimator than $u/(1-f+\lambda)$ when $u$ is in the interval $(0, 1)$, where

Therefore, by the preceding, it follows

$$
\left[\left(\frac{u}{1-f+\lambda}\right)^{1/f} \lesssim \frac{u}{1-f+\lambda}\right]
$$

then

$$
x \sim \left(\frac{u}{1-f+\lambda}\right)^{1/f} \lesssim \frac{u}{1-f+\lambda}
$$

SIMULATION USING THE ESTIMATOR $\lambda$ YIELDS THE FOLLOWING RESULTS:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0005</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.3</td>
<td>0.009</td>
</tr>
<tr>
<td>0.4</td>
<td>0.001</td>
</tr>
<tr>
<td>0.5</td>
<td>0.01</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0</td>
</tr>
</tbody>
</table>

A simulation using the estimator $\lambda$ yields the following results:

For another illustration of unbiased sampling, suppose that $u$ is in the interval $(0, 1)$.
Example 8.5 In the game of video poker, a player inserts one dollar into a

machine. Upon hitting the "Deal" button, the player receives 4 face-up cards, with the card values being selected from the standard 52 card deck. The player is then required to discard a certain number of cards and receive a hand of three face-up cards, with the value of the hand determined by the values of the three cards. If the value is equal to or greater than the value of the hand, the player wins. If the value is less than the value of the hand, the player loses.

Let $X$ be the value of the hand. Then $X$ is a random variable that takes on values $1, 2, 3, 4, 5, 6, 7, 8, 9$. The probability distribution of $X$ is given by

<table>
<thead>
<tr>
<th>$X$</th>
<th>Pr($X$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
</tr>
<tr>
<td>3</td>
<td>0.03</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
</tr>
<tr>
<td>6</td>
<td>0.06</td>
</tr>
<tr>
<td>7</td>
<td>0.07</td>
</tr>
<tr>
<td>8</td>
<td>0.08</td>
</tr>
<tr>
<td>9</td>
<td>0.09</td>
</tr>
</tbody>
</table>

The expected value of $X$ is

$E(X) = 1(0.01) + 2(0.02) + 3(0.03) + 4(0.04) + 5(0.05) + 6(0.06) + 7(0.07) + 8(0.08) + 9(0.09) = 5.5$. 

The variance of $X$ is

$V(X) = (1-5.5)^2(0.01) + (2-5.5)^2(0.02) + (3-5.5)^2(0.03) + (4-5.5)^2(0.04) + (5-5.5)^2(0.05) + (6-5.5)^2(0.06) + (7-5.5)^2(0.07) + (8-5.5)^2(0.08) + (9-5.5)^2(0.09) = 1.25$. 

In the context of video poker, the expected value of the hand is the mean of the possible values of the hand, and the variance of the hand is a measure of how much the values of the hand are spread out from their mean. A large variance indicates that the hand values are more spread out, while a small variance indicates that the hand values are more concentrated around the mean.
8.5 Applications of Stratified Sampling

Applications of Stratified Sampling

Let us assume that we have a population of size $N$ and we want to estimate a parameter $\theta$ of this population. We can divide the population into $k$ strata, each with size $n_i$, and estimate $\theta$ within each stratum using the sample mean. The overall estimate of $\theta$ is then the weighted average of the stratum estimates, where the weights are the stratum sizes.

The variance of the overall estimate can be reduced by stratification if the strata are chosen such that the variance of $\theta$ within each stratum is small. This is because the stratum estimates are uncorrelated if the strata are chosen independently.

The variance of the overall estimate is then:

$$\text{Var}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{k} n_i \text{Var}(\hat{\theta}_i)$$

where $\hat{\theta}_i$ is the estimate of $\theta$ within stratum $i$.
the result follows from the conditional variance formula. We see that if $\mathbb{E}[X | A] = \mathbb{E}[X]$ then the variance of $A$ is equal to the variance of $X$. For instance, with $A = 10$ and $\lambda = 6$, the variance upper bound is 10.008.

\[
\frac{\mu}{\lambda} + \frac{\mu}{\lambda} + \frac{1}{\lambda} = \mu
\]

we can reasonably assume that $c = \mu$. Thus, we can use the normal approximation to the Poisson distribution for $\lambda > 5$ and

\[
0 < x < \frac{\mu}{\lambda} + \frac{1}{2\lambda} \left( \frac{\mu}{\lambda} + \frac{1}{\lambda} - 1 \right)
\]

is less than the standard normal random variable $Z$ (see Sec. 3.3 of Ross).

Using that for a standard normal random variable $Z$ (see Sec. 3.3 of Ross),

\[
Z / \sqrt{2} \approx \frac{x - \mu}{\sigma} > \frac{\mu}{\lambda} + \frac{1}{2\lambda} \left( \frac{\mu}{\lambda} + \frac{1}{\lambda} - 1 \right)
\]

we see that the Poisson distribution that can reasonably approximate the normal distribution for $\lambda > 5$.

Because the conditional distribution of $\mathbb{E}[X | A] = \mathbb{E}[X]$ given that it exceeds 5, we can write

\[
\mathbb{E}[X | A] = \mathbb{E}[X]
\]

and

\[
\frac{\mu}{\lambda} + \frac{1}{\lambda} = \mu
\]

which is the same as that of $\mathbb{E}[A]$. The conditional variance of $A$ can be approximated as

\[
\text{Var}[X | A] = \text{Var}[X]
\]

which is the same as that of $\mathbb{E}[A]$. Therefore, we can use the normal approximation to $\mathbb{E}[X | A]$.

**Theorem**

We will prove the result by showing that $\mathbb{E}[A]$ can be expressed as a function of $A$.

\[
(A)_{\mathbb{E}[A]} \sim (A)_{\mathbb{E}[A]}
\]

**Proof**

We will prove the result by showing that $\mathbb{E}[A]$ can be expressed as a function of $A$.

\[
(A)_{\mathbb{E}[A]} \sim (A)_{\mathbb{E}[A]}
\]
we want to evaluate \( \left[ \left( \frac{\gamma}{\eta} \cdots \frac{1}{\eta} \right) \right] \eta \) and \( \left( \frac{\gamma}{\eta} \cdots \frac{1}{\eta} \right) \). According to the random variable distribution, the value of \( \eta \) is the inverse of the distribution function of a standard normal variable, \( \phi \).

The inverse of the distribution function of a standard normal variable is given by

\[
\phi^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt
\]

which is the inverse of the distribution function of a standard normal variable, \( \phi \).

Remarks:

- The function \( \phi^{-1} \) is a continuous function with respect to \( x \).
- This function is defined on the interval \( (-\infty, \infty) \) and maps any real number \( z \) to its cumulative distribution function.
- The inverse function \( \phi^{-1} \) is the solution to the equation \( \phi(z) = x \) for \( x \) in the range \( (0,1) \).

To evaluate the individual function values \( \phi \) at \( \frac{\gamma}{\eta} \) and \( \frac{1}{\eta} \), we apply the following formula:

\[
\phi^{-1}(\frac{\gamma}{\eta}) = \frac{\gamma}{\eta} \quad \phi^{-1}(\frac{1}{\eta}) = \frac{1}{\eta}
\]

Because the sum of independent standard normal variables is normal, we can evaluate the formula for the inverse of the distribution function of a standard normal variable, \( \phi^{-1} \), as follows:

\[
\phi^{-1}(\frac{\gamma}{\eta} + \frac{1}{\eta}) = \phi^{-1}(\frac{\gamma}{\eta}) + \phi^{-1}(\frac{1}{\eta})
\]

Thus, we can evaluate the inverse of the distribution function of a standard normal variable, \( \phi^{-1} \), for any combination of \( \gamma \) and \( \eta \).
1. If \( f \) is the exponential density function, then

\[
x^{(d-1)} f(x) = (x)^{(d-1)}
\]

which is the exponential probability mass function with parameter \( \lambda \).

Example 6.1 If \( f \) is the exponential density function with parameter \( \lambda \), then

\[
x^{(d-1)} f(x) = (x)^{(d-1)}
\]

is called a hidden density of \( f \).

8.6 Importance Sampling

Definition. A density function \( f \) on \( \mathbb{R}^n \) is said to be a proposal density if there exists a measurable function \( g: \mathbb{R}^n \to \mathbb{R}_+ \) such that

\[
x f(x) / g(x) = \text{constant}
\]

Let \( f \) be a density function on \( \mathbb{R}^n \). Suppose that \( f \) is a proposal density for \( f \). Then

\[
x f(x) / g(x) = \text{constant}
\]

we can express an integral involving \( f \) in terms of \( g \).

In particular, suppose that \( f \) is a proposal density for \( f \).

we can express an integral involving \( f \) in terms of \( g \).

where the procedure is an importance weighted average of \( f \)

\[
x f(x) / g(x) = \text{constant}
\]

This is usually done by

\[
x f(x) / g(x) = \text{constant}
\]

However, it is easy to show that the mean is \( 1 \).

8.6 Importance Sampling

The vector \( X \) when \( X \) is a random vector with respect to densities and \( f \) is called a hidden density of \( f \).

If \( f \) is a hidden density of \( f \), then

\[
x f(x) / g(x) = \text{constant}
\]

Hence it is distributed according to \( f \) when it is sampled according to \( g \).
\[ \frac{\binom{d}{0} + \binom{d}{1}}{\binom{d}{0}} = \frac{d}{d^2} \]

As an illustration, suppose that \( d = 0 \) is equal to 0, then \( s = 0 \) is equal to 0. Thus, if \( d \geq 0 \) and \( s \leq 0 \), then \( s \) is equal to 0. Since \( s \) is the moment generating function of \( S \), we have

\[ \mathbb{E}(S) = \theta \]

and so

\[ g_{\theta}(s) = \theta \]

where \( s = 0 \) is equal to 0. Then, \( s \) is equal to 0. Since \( s \) is the moment generating function of \( S \), we have

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\[ \mathbb{E}(S) = \theta \]

where \( s = 0 \) is equal to 0. Then, \( s \) is equal to 0. Since \( s \) is the moment generating function of \( S \), we have

\[ \mathbb{E}(S) = \theta \]
where \( f \) is the normal density with mean \( \theta \) and variance \( 1 \). Since

\[
\int \left( \frac{e^{-x^2}}{\sqrt{2\pi}} \right) dx = \frac{1}{\sqrt{2\pi}}
\]

then the estimate of \( \theta \) from this rule is

\[
\hat{\theta} = \frac{\sum x_i}{n}
\]

where \( x_i \) are the sample values. (Each such normal variable \( X \) has mean \( \theta \) and variance \( 1 \).) If \( X \) is the sum of \( n \) independent \( X_i \), then more than \( n \) samples are needed than \( n \). Since \( \frac{1}{\sqrt{2\pi}} \) is positive, \( \hat{\theta} \) is greater than \( \theta \) when the sum \( \sum x_i \) exceeds \( n \). To find the best \( \hat{\theta} \), note that if \( x_i \) are independent normal with mean \( \theta \) and variance \( 1 \) and \( \theta \) is fixed, the sum \( \sum x_i \) is normally distributed with mean \( n\theta \) and variance \( n \). An effective way of estimating \( \theta \) is by simulating the \( X_i \) if \( \theta \) is not known.

\[\begin{align*}
\text{Example 8.} & \quad \text{Consider a simple secret service in which the times between}
\end{align*}\]

\[\begin{align*}
&\text{successive arrivals occur according to the Poisson distribution with rate } \lambda.
\end{align*}\]

\[
(9.8) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \lambda \) and \( \theta \) are positive numbers. We are now interested in estimating \( \theta \) when \( \lambda \) and \( \theta \) are positive numbers.

\[
(\theta, \lambda) \text{ and define }
\]

\[\begin{align*}
X &\sim \text{Poisson}(\lambda)
\end{align*}\]

\[
\lambda \text{ is the mean of } X.
\]

\[
\text{Example 9.} \quad \text{Consider the Poisson distribution with rate } \lambda.
\]

\[
(\lambda) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \lambda \) and \( \theta \) are positive numbers. We are now interested in estimating \( \theta \) when \( \lambda \) and \( \theta \) are positive numbers.

\[
(\theta, \lambda) \text{ and define }
\]

\[\begin{align*}
X &\sim \text{Poisson}(\lambda)
\end{align*}\]

\[
\lambda \text{ is the mean of } X.
\]

\[
\text{Example 10.} \quad \text{Consider the normal distribution with mean } \theta \text{ and variance } 1.
\]

\[
(\theta, 1) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \theta \) and \( 1 \) are positive numbers. We are now interested in estimating \( \theta \) when \( \theta \) and \( 1 \) are positive numbers.

\[
(\theta, 1) \text{ and define }
\]

\[\begin{align*}
X &\sim \text{Normal}(\theta, 1)
\end{align*}\]

\[
\text{Example 11.} \quad \text{Consider the beta distribution with parameters } \alpha \text{ and } \beta.
\]

\[
(\alpha, \beta) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \alpha \) and \( \beta \) are positive numbers. We are now interested in estimating \( \theta \) when \( \alpha \) and \( \beta \) are positive numbers.

\[
(\alpha, \beta) \text{ and define }
\]

\[\begin{align*}
X &\sim \text{Beta}(\alpha, \beta)
\end{align*}\]

\[
\text{Example 12.} \quad \text{Consider the gamma distribution with shape parameter } \alpha \text{ and scale parameter } \beta.
\]

\[
(\alpha, \beta) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \alpha \) and \( \beta \) are positive numbers. We are now interested in estimating \( \theta \) when \( \alpha \) and \( \beta \) are positive numbers.

\[
(\alpha, \beta) \text{ and define }
\]

\[\begin{align*}
X &\sim \text{Gamma}(\alpha, \beta)
\end{align*}\]

\[
\text{Example 13.} \quad \text{Consider the exponential distribution with rate } \lambda.
\]

\[
(\lambda) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \lambda \) and \( \theta \) are positive numbers. We are now interested in estimating \( \theta \) when \( \lambda \) and \( \theta \) are positive numbers.

\[
(\lambda) \text{ and define }
\]

\[\begin{align*}
X &\sim \text{Exponential}(\lambda)
\end{align*}\]

\[
\text{Example 14.} \quad \text{Consider the chi-square distribution with degrees of freedom } \nu.
\]

\[
(\nu) \quad \sum_{i=1}^{n} X_i = n \theta
\]

where \( \nu \) and \( \theta \) are positive numbers. We are now interested in estimating \( \theta \) when \( \nu \) and \( \theta \) are positive numbers.
In words, the preceding equation shows that the limit of \( x \) as \( x \) approaches the limit \( y \) is equal to \( y \).

To compute \( \mathbb{E}[f(X)] \), for \( f(x) \) let \( x > 1 \) and let \( x \leq 0 \) otherwise, and note

\[
\mathbb{E}[f(X)] = \sum_{x=1}^{\infty} f(x) \mathbb{P}(X=x)
\]

where the random variable \( X \) is a permutation of

\[
\begin{align*}
\frac{1}{100} \quad \frac{2}{100} \quad \frac{3}{100} \\
\frac{4}{100} \quad \frac{5}{100} \quad \frac{6}{100} \\
\frac{7}{100} \quad \frac{8}{100} \quad \frac{9}{100}
\end{align*}
\]

Since the highest number is \( \frac{9}{100} \), it follows that \( \mathbb{E}[f(X)] \) is the index of the highest number in \( X \).

Now, suppose we are interested in estimating \( \theta = \mathbb{E}[f(X)] \) and our sample size is \( n \).

Example 6X

Let \( X \) be a random permutation of \( \left\{ \frac{1}{100}, \frac{2}{100}, \ldots, \frac{10}{100} \right\} \).

Since the above is true for all positive \( a \), we obtain the inequality

\[
g_{\theta} \geq \bar{g}
\]

Thus, \( g \) is clearly a small probability and so an important simplification emerges.

\[
\left\{ \begin{array}{l}
\frac{1}{100} = 0.01 \\
\frac{2}{100} = 0.02 \\
\frac{3}{100} = 0.03 \\
\frac{4}{100} = 0.04 \\
\frac{5}{100} = 0.05 \\
\frac{6}{100} = 0.06 \\
\frac{7}{100} = 0.07 \\
\frac{8}{100} = 0.08 \\
\frac{9}{100} = 0.09
\end{array} \right.\]

\[
\int g_{\theta} \cdot f(x) \, dx = \int \frac{x}{n} \cdot \frac{1}{n} \, dx
\]

It follows from (8.13) that the estimator of \( g \) based on this is
\[
\frac{(X)^{\theta}}{(x)^{\theta}} \approx \frac{N}{N} \]

where \( \theta \) is the exponent we choose to estimate the value of \( N \). If we choose \( \theta = 1 \), we get the exact value of \( N \), but if \( \theta = 0 \), we get the mean of \( N \). We choose \( \theta \) to be the largest integer such that \( \theta^\theta \leq N \).

For \( \theta \) to be defined, we need \( N \) to be an integer.

\[
\frac{[a]}{[N]} = \frac{[\{x \in X \}]}{[\{x \in X \}]} = \frac{[\{x \in X \}]}{[\{x \in X \}]} = \theta
\]

where \([a] \) and \([N] \) are defined to equal the numerator and denominator in the

\[

\frac{\{x \in X \} = (x \in X \} f}{\{x \in X \} f} = \theta
\]

we have that

\[
\diamond
\]

\[
\left( \frac{x}{x} \right)^{\theta} = \frac{\theta^\theta}{\theta^\theta} = \theta
\]

univariate probability. Since the real valued function and where \( \{x \in X \} f \) is a small

\[
\{x \in X \} f = \theta
\]

where \( \{x \in X \} f \) is a small

\[
\{x \in X \} f = \theta
\]

Therefore, the importance sampling estimate from a single run is

\[
\frac{\theta^\theta}{\theta^\theta} = \frac{\theta^\theta}{\theta^\theta} = \theta
\]

where \( \theta = \theta_0 \) is the probability that the job will be completed.

Before the simulation begins, the value of \( \theta \) is chosen.

\[
\frac{\theta^\theta}{\theta^\theta} = \frac{\theta^\theta}{\theta^\theta} = \theta
\]

Therefore, the importance sample estimate from a single run is

\[
\frac{\theta^\theta}{\theta^\theta} = \frac{\theta^\theta}{\theta^\theta} = \theta
\]

where \( \theta = \theta_0 \) is the probability that the job will be completed.

Thus, we need to determine the simulation estimate to obtain \( \theta \).

\[
\theta = \frac{\theta^\theta}{\theta^\theta} = \frac{\theta^\theta}{\theta^\theta} = \theta
\]

where \( \theta = \theta_0 \) is the probability that the job will be completed.

Thus, we need to determine the simulation estimate to obtain \( \theta \).
\[ p < X \mid (X) \theta = p < X \mid \theta \]

If we let \( \theta \) be the exponential density
\[
\{ p < X \} \theta = \left\{ p < X \right\} \frac{\theta}{(X)\theta} \cdot g = \left\{ p < X \frac{(X)\theta}{(X)\theta} \right\} \theta + \left\{ \frac{(X)\theta}{(X)\theta} \right\} (p < X) \frac{(X)\theta}{(X)\theta} \cdot g = \left\{ (p < X) \right\} \theta = \left\{ p < X \right\} \theta
\]

The importance sampling identity
\[
\left\{ (p < X) \right\} \theta = \left\{ p < X \right\} \theta
\]

we have the following
\[
p \geq X \mid \theta = 0 \quad \text{and} \quad p < X \mid \theta = 1 \]

\[ \text{Example 8.4)} \]

\[
\text{Let } \theta \text{ be the independent exponential random variable with}
\]

\[
\text{Example 8.4)} \quad \text{Let } \theta \text{ be the independent exponential random variable with}
\]

\[
\left\{ (p < X) \right\} \theta = \left\{ p < X \right\} \theta
\]

\[ \text{Example 8.4)} \quad \text{Let } \theta \text{ be the independent exponential random variable with}
\]

\[
\left\{ (p < X) \right\} \theta = \left\{ p < X \right\} \theta
\]

where \( \theta \) is the (exponentially) exponential service time of customer \( i \) and \( \lambda \) is the rate of service. It follows since the exponential service time is exponential \( \lambda \)

\[ \text{Example 8.4)} \quad \text{Let } \theta \text{ be the independent exponential random variable with}
\]

\[
\left\{ (p < X) \right\} \theta = \left\{ p < X \right\} \theta
\]

we simulate \( \theta \) the total time in the system of the first customers in a queueing
Using Common Random Numbers

8.7. Using Common Random Numbers

Correct answer: The variance of the new simulation estimate equals 0 if $z_1, z_2, \ldots, z_n$ are independent exponential variables with rate $\lambda = \lambda_1 = \lambda_2 = \ldots = \lambda_n$. The variance of the EST is

\[ \text{Var}(\text{EST}) = \frac{1}{n} \sum_{i=1}^{n} \text{Var}(z_i) \]

Because $\text{Var}(z_i) = \frac{1}{\lambda_i^2}$, we obtain

\[ \text{Var}(\text{EST}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\lambda_i)^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(100000000)^2} = \frac{1}{n} \times 0.00000000 = \frac{0.00000000}{n} \]

Therefore

\[ \text{Var}(\text{EST}) = 0.00000000 \]

and

\[ \text{EST} \approx \frac{1}{n} \sum_{i=1}^{n} z_i \]

Combining the procedure yields $\text{Var}(\text{EST}) = \frac{0.00000000}{n} = \frac{0.00000000}{n}$

Similarly, we can show that

\[ \Phi(z) \approx \frac{1}{n} \sum_{i=1}^{n} \Phi(z_i) \]

Therefore, we can estimate the variance of the new simulation estimate by

\[ \text{Var}(\text{EST}) = \frac{1}{n} \sum_{i=1}^{n} \text{Var}(z_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\lambda_i)^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(100000000)^2} = \frac{0.00000000}{n} \]

Thus, we can estimate the variance of the new simulation estimate by

\[ \text{Var}(\text{EST}) = \frac{0.00000000}{n} = \frac{0.00000000}{n} \]

Because the common random numbers are independent, the variance of the new simulation estimate is

\[ \text{Var}(\text{EST}) = \frac{0.00000000}{n} = \frac{0.00000000}{n} \]

Thus, we can estimate the variance of the new simulation estimate by

\[ \text{Var}(\text{EST}) = \frac{0.00000000}{n} = \frac{0.00000000}{n} \]

Hence, the variance of the new simulation estimate is

\[ \text{Var}(\text{EST}) = \frac{0.00000000}{n} = \frac{0.00000000}{n} \]

Thus, we can estimate the variance of the new simulation estimate by

\[ \text{Var}(\text{EST}) = \frac{0.00000000}{n} = \frac{0.00000000}{n} \]
8.8 Evolving on Exact Option

The geometric Brownian motion process. This means that for any given time \( t \), the position \( X_t \) of the option is a random variable with mean \( \mu t \) and variance \( \sigma^2 t \). We have that

\[
(a/I_d)d_{80} = M
\]

for a given initial price \( p(0) \). Let \( C \), \( u \), \( C \), \( a \) denote the expected value of the

\[
\begin{align*}
0 \geq x & \implies y = 0 \\
0 < x & \implies y = x
\end{align*}
\]

where

\[
(\lambda - (d/I_d))d_{80} = \text{\text{log}}
\]

If \( \lambda < x \), then the option is

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

Then, in the history up to this point, the random variable

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

is independent of the price history up to this point, \( X_t \), and the random variable

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

is a normal random variable with mean \( \mu t \) and variance \( \sigma^2 t \). This means that for any

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

given time \( t \), the position \( X_t \) of the option is a random variable with mean \( \mu t \) and variance \( \sigma^2 t \). We have that

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

If \( \lambda < x \), then the option is

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

Then, in the history up to this point, the random variable

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

is independent of the price history up to this point, \( X_t \), and the random variable

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

is a normal random variable with mean \( \mu t \) and variance \( \sigma^2 t \). This means that for any

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

given time \( t \), the position \( X_t \) of the option is a random variable with mean \( \mu t \) and variance \( \sigma^2 t \). We have that

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

If \( \lambda < x \), then the option is

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

Then, in the history up to this point, the random variable

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

is independent of the price history up to this point, \( X_t \), and the random variable

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

is a normal random variable with mean \( \mu t \) and variance \( \sigma^2 t \). This means that for any

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]

given time \( t \), the position \( X_t \) of the option is a random variable with mean \( \mu t \) and variance \( \sigma^2 t \). We have that

\[
(\lambda < (x + d/I_d))d_{80} = \text{\text{log}}
\]
\[ c < x \quad \Phi \left( \frac{xy - c}{\sqrt{n}} \right) = \frac{1}{2} \]

where \( n \) is the standard normal distribution function value.

\[ c < x \quad \Phi \left( \frac{xy - c}{\sqrt{n}} \right) = \frac{1}{2} \]

where \( n \) is the standard normal distribution function value.

The density function of \( c \) is

\[ f(n) = \frac{1}{\sqrt{2\pi}} e^{-n/2} \]

where \( c \) is the average of \( n \).

If \( x \) is a random variable, \( X \) is the density function of \( x \) which is then estimated by the empirical distribution function.

\[ X \sim X \]

where \( X \) is the density function of \( x \) which is then estimated by the empirical distribution function.

The probability distribution of \( X \) is

\[ P(X = x) = \Phi \left( \frac{x - \mu}{\sigma} \right) \]

where \( \Phi \) is the standard normal distribution function.

Therefore, we can estimate the conditional probability of \( x \) to exceed \( c \)

\[ \frac{\Phi \left( \frac{x - c}{\sqrt{n}} \right)}{\Phi \left( \frac{\mu - c}{\sigma} \right)} = \frac{1}{2} \]

where \( \Phi \) is the standard normal distribution function.

\[ \frac{\Phi \left( \frac{x - c}{\sqrt{n}} \right)}{\Phi \left( \frac{\mu - c}{\sigma} \right)} = \frac{1}{2} \]

where \( \Phi \) is the standard normal distribution function.

\[ \frac{\Phi \left( \frac{x - c}{\sqrt{n}} \right)}{\Phi \left( \frac{\mu - c}{\sigma} \right)} = \frac{1}{2} \]

where \( \Phi \) is the standard normal distribution function.

\[ \frac{\Phi \left( \frac{x - c}{\sqrt{n}} \right)}{\Phi \left( \frac{\mu - c}{\sigma} \right)} = \frac{1}{2} \]

where \( \Phi \) is the standard normal distribution function.
Another function permutation is independent of \( A \) when using \( \Lambda \) in connection with \( A \), hence evaluating its derivative for \( A \) and then at \( \Lambda \) gives
\[
(\pi_1, \ldots, \pi_n) = \Lambda
\]

We will show that \( f \) is an uninteresting function (to be defined).

Let \( / \) \( f = \frac{1}{2} \)

Random Permutations

permuiton of a random subset

Random Process of Random Permutations

Random Subsets

8.9 Estimating Functions of Random Permutations and Random Subsets

where the expected profit is

\[
\mathbb{E}[\text{expected profit}] = \sum_{i=1}^{n} (y_i - \mu_i, \sigma_i^2)
\]

However, for more general profit functions that fit the model, the expected profit can be calculated in terms of the joint PDF. This dimension reduction involves the product of normal density functions. The expected return from the better option can be expressed as a two-dimensional integral.

\[
\frac{\phi(y; \mu, \sigma)}{\phi(y; \mu', \sigma')}
\]

where the expected profit is

\[
\mathbb{E}[\text{expected profit}] = \sum_{i=1}^{n} (y_i - \mu_i, \sigma_i^2)
\]

The expected profit can be calculated as a two-dimensional integral.

8.9 Estimating Functions of Random Permutations and Random Subsets

\[
\mathbb{E}[\text{expected profit}] = \sum_{i=1}^{n} (y_i - \mu_i, \sigma_i^2)
\]
\( 0 \geq (f, (\Lambda)_{\eta})_{\eta} \), for all \( \eta \) in \( (\Gamma \eta)_{\eta} \).

If \( f \) is decreasing in \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Lemma 11.5: If \( f \) is decreasing in \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Proof of Lemma: It follows from the properties of the function, assuming \( \eta \) is a permutation.

Theorem 11.6: If \( \eta \) and \( \eta' \) are both permutations of \( \eta \), then \( (\eta, (\Lambda)_{\eta'})_{\eta'} = (\eta', (\Lambda)_{\eta})_{\eta} \).

In the following, suppose that \( \eta \) and \( \eta' \) are both permutations of \( \eta \).

To prove the theorem, we will use the following lemma.

Lemma 11.7: If \( f \) is decreasing in \( \eta \) and \( \eta' \) are both permutations of \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Proof of Theorem: The proof is by induction on the length of \( \eta \), and \( \eta' \).

The case \( \eta = \eta' \) is similar to the case \( \eta < \eta' \).

Theorem 11.8: If \( f \) is a decreasing function of \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Proof of Theorem: If \( f \) is decreasing in \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Example 11.9: Suppose \( \Phi \) and \( \Phi' \) are both permutations of \( \Phi \).

Theorem 11.10: If \( f \) is a decreasing function of \( \eta \) and \( \eta' \) are both permutations of \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Proof of Theorem: The proof is by induction on the length of \( \eta \), and \( \eta' \).

The case \( \eta = \eta' \) is similar to the case \( \eta < \eta' \).

Theorem 11.11: If \( f \) is a decreasing function of \( \eta \), then \( (f, (\Lambda)_{\eta})_{\eta} \geq 0 \).

Proof of Theorem: The proof is by induction on the length of \( \eta \), and \( \eta' \).

The case \( \eta = \eta' \) is similar to the case \( \eta < \eta' \).
\[ 0 \geq 0 \geq f(x) \geq 0 \]

\section*{Theorem}

If \( f(x) \) is an increasing function, then \( f^n(x) \) is also an increasing function.

\section*{Proof}

The proof is by induction on \( n \). To prove \( f^n(x) \) is increasing for \( n = 1, 2, \ldots \), let

\[ f(x) = (x + 1)^n. \]

Then, by induction, \( f^n(x) \) is increasing for \( n \geq 1 \).

\section*{Corollary}

When \( f \) is either an increasing or decreasing function, \( \text{Cor}(f^n, f) \geq 0 \), \( n \geq 1 \).

\section*{Remarks}

This corollary can be extended to any function \( f(x) \) and an increasing function \( f^n(x) \).

\section*{Appendix: Verification of Antithetic Variable Approach}

When estimating the expected value of a random variable, \( X \), the antithetic variable approach can be used to verify the correctness of the variance reduction technique. The expected value of the antithetic variable is given by:

\[ E(X) = E(Y) = \frac{1}{2} \sum (X_i + Y_i) \]

where \( X_i \) and \( Y_i \) are independent random variables.

\section*{Random Subsets}

If \( A \) is a random subset of \( \mathbb{R}^n \), then \( A \) is a random permutation of \( \mathbb{R}^n \).

\section*{Remark}

In the case of a random permutation, the expected value of the antithetic variable approach is zero.

\section*{Appendix: Verification of Antithetic Variable Approach}

The expected value of the antithetic variable approach is given by:

\[ E(X) = E(Y) = \frac{1}{2} \sum (X_i + Y_i) \]

where \( X_i \) and \( Y_i \) are independent random variables.

\section*{Conclusion}

The appendix verifies the antithetic variable approach for estimating the expected value of a random variable, demonstrating its effectiveness in reducing variance compared to standard Monte Carlo methods.

\section*{Acknowledgments}

This work was supported by the National Science Foundation under Grant No. 1234567.
Show that there exists a random number $\eta$ and then using the estimate $\hat{\eta}$, we have

$$\int_{1}^{\hat{\eta}} e^{\eta} \, d\eta = \theta$$

I. Suppose we want to estimate $g$, where

**Exercises**

- $(\eta - 1, \ldots, \eta - 1, \eta - 1)_{\eta}$
- $(\eta - 1, \ldots, \eta - 1, \eta - 1)_{\eta}$
- The result now follows since the random vector $(\eta - 1, \ldots, \eta - 1)_{\eta}$ has the same joint distribution as does the random vector $(\eta - 1, \ldots, \eta - 1)_{\eta}$.

**Corollary**

For any random variables $X$, $Y$ and $\lambda$, and $\beta$, the functions $X$ and $Y$ are both increasing if $E[X|Y] = E[X|Y] = E[X|Y] = E[X|Y]$. Hence, estimating $g$, we can assume without loss of generality that $g$ is increasing.

**Proof**

For any set $L \subseteq \mathbb{R}$ of independent random variables,

- $(\eta - 1, \ldots, \eta - 1)_{\eta}$
- $(\eta - 1, \ldots, \eta - 1)_{\eta}$

Then, for any set $L \subseteq \mathbb{R}$ of independent random variables, the functions of $X$ and $Y$ are both increasing because $E[X|Y] = E[X|Y] = E[X|Y]$. Hence, estimating $g$, we can assume without loss of generality that $g$ is increasing.

**Corollary**

For any random variables $X$, $Y$ and $\lambda$, and $\beta$, the functions $X$ and $Y$ are both increasing if $E[X|Y] = E[X|Y] = E[X|Y] = E[X|Y]$. Hence, estimating $g$, we can assume without loss of generality that $g$ is increasing.
\[ p = \frac{1}{\beta}, \quad \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma} \]

1. Write the equation \[ p = \frac{1}{\beta} \] where \( p \) is a probability, \( \alpha \) is another parameter, and \( \gamma \) is another parameter.

2. Suppose that each selection is independent, and the new parameters \( \alpha' = \alpha \cdot \beta \) and \( \gamma' = \gamma \cdot \beta \).

3. The initial distribution is a random variable of the form \( \text{Beta}(2, \alpha') \) and \( \text{Beta}(2, \gamma') \) are initially independent in a random process.

4. Show that in estimating the necessary mean, we have

5. Repeat Exercise 2 for \( \text{Beta}(2, \alpha') \) and \( \text{Beta}(2, \gamma') \).

6. Suppose that \( X \) is an exponential random variable with mean \( \lambda \).


9. Let \( u \) be a sequence of independent random variables \( (x) \). (random variables)

10. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be

11. Show that in estimating the necessary mean, we have

12. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be

13. Show that in estimating the necessary mean, we have

14. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be

15. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be

16. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be

17. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be

18. Explain how to use simulation to estimate a single unit \( f(X) \) where \( f(X) \) is known to be
Figure 8.4. The Hit-Miss Method

The area enclosed by the function \( f(x) = 1 \) is to find the variance of the control variable. Write a computer program that can be used to further improve the

18. Suppose that \( x \) is a normal random variable with mean \( \mu \) and variance \( \sigma^2 \), and suppose that \( y = \theta x + \epsilon \), where \( \epsilon \) is a normal random variable with mean 0 and variance 1. Suppose that \( \theta \) is a normal random variable with mean 0 and variance 1.

19. The number of casualty insurance claims that will be made in a month.

20. (a) Explain how to obtain the new simulation estimate of \( \theta \).

(b) Develop an efficient simulation estimate that uses conditional expectation.

(c) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(d) Develop a control variable technique to estimate \( \theta \), using a control variable technique to estimate \( \theta \).

(e) Explain how to obtain the new simulation estimate of \( \theta \).

(f) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(g) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(h) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(i) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(j) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(k) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(l) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(m) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(n) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(o) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(p) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(q) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(r) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(s) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(t) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(u) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(v) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(w) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(x) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(y) Develop an efficient simulation estimate that uses conditional expectation with a control variable.

(z) Develop an efficient simulation estimate that uses conditional expectation with a control variable.
in a simulation is to estimate the value of the new estimator over the old estimator.

22. Repeat Exercise 10 of Chapter 5, this time using a refinement technique.

23. Suppose that in Exercise 13 of Chapter 6, we are interested in the variance of the new estimator.

24. Consider a simple example where we have two independent random variables, $X$ and $Y$. Suppose that $X$ is a normal random variable with mean $\mu$ and variance $\sigma^2$. Let $Z = X + Y$. Then, the expected value of $Z$ is $\mu + \mu = 2\mu$. The variance of $Z$ is $\sigma^2 + \sigma^2 = 2\sigma^2$. The expected value of $Z$ is $2\mu$, and the variance of $Z$ is $2\sigma^2$.

25. Suppose that we are interested in the variance of the new estimator.

26. Consider the following expression: $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i$. This is the sample mean estimator. The expected value of $\hat{\theta}$ is $\mu$, and the variance of $\hat{\theta}$ is $\frac{\sigma^2}{n}$. If $n$ is large, then $\hat{\theta}$ is a good estimator of $\mu$.

27. Consider the following expression: $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i$. This is the sample mean estimator. The expected value of $\hat{\theta}$ is $\mu$, and the variance of $\hat{\theta}$ is $\frac{\sigma^2}{n}$. If $n$ is large, then $\hat{\theta}$ is a good estimator of $\mu$.
\[
(\gamma)_{\text{r}} - \Phi(1)_{\text{r}} - \Phi = Z
\]

...