This laboratory session is focused on modelling, analysis and numerical simulation of the mechanical system shown in Fig. 1, which consists of an inverted pendulum mounted on a cart (cart–pendulum system).

![Cart–pendulum system](image)

**Figure 1:** Cart–pendulum system

### Exercise 1: simulation of the nonlinear cart–pendulum model

Consider the nonlinear dynamical model of the cart–pendulum system derived in class

$$\Sigma_{nl} : \begin{cases} I_e \ddot{\theta} - ml \cos \theta \ddot{x} - mg l \sin \theta = 0 \\ -ml \cos \theta \ddot{\theta} + M_e \ddot{x} + b \dot{x} + ml \sin \theta \dot{\theta}^2 = F \end{cases}$$

with $I_e \triangleq I + ml^2$ and $M_e \triangleq m + M$.

1. Implement a Simulink model to simulate the nonlinear dynamics (1)–(2) of the cart–pendulum system, by using the modelling approach based on chain of integrators. This approach consists of rewriting (1)–(2) as follows

$$\ddot{\theta} = f_p(x, \dot{x}, \ddot{x}, \theta, \dot{\theta}) = (ml \cos \theta \ddot{x} + mg l \sin \theta) / I_e$$

$$\ddot{x} = f_c(x, \dot{x}, \theta, \dot{\theta}, \dot{\theta}, F) = \left( ml \cos \theta \ddot{\theta} - b \dot{x} - ml \sin \theta \dot{\theta}^2 + F \right) / M_e$$

and then implementing (3) and (4) as shown in Fig. 2. In the Simulink model shown in figure, the Pendulum dynamics (nonlinear) and Cart dynamics (nonlinear) blocks are two Simulink → User-Defined Functions → Fcn blocks that implement the two functions $f_p$ and $f_c$ defined in (3)–(4).

The Signal Routing → From and Signal Routing → Goto blocks are useful for connecting blocks without tracing too many wires in the Simulink model (which may cause confusion and errors).

![Simulink implementation of the cart–pendulum nonlinear dynamics](image)

**Figure 2:** Possible Simulink implementation of the cart–pendulum nonlinear dynamics.

### Table 1: Nominal parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Cart mass</td>
<td>$M$</td>
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<tr>
<td>Pendulum mass</td>
<td>$m$</td>
</tr>
<tr>
<td>Pendulum half-length</td>
<td>$l$</td>
</tr>
<tr>
<td>Pendulum moment of inertia</td>
<td>$I = \frac{1}{2} ml^2$</td>
</tr>
<tr>
<td>Cart viscous friction coefficient</td>
<td>$b$</td>
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</table>

**Pendulum dynamics (…) → Expression field:**

$$(m*1/Ie) * ( \cos(u(1)) * u(2) + g*\sin(u(1)))$$

**Cart dynamics (…) → Expression field:**

$$1/Me * (-b*u(1) + m*l*\cos(u(2))*u(4) - m*l*\sin(u(2))*u(3)^2 + u(5))$$

**Cart-pendulum nonlinear dynamics**

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2. Denote with \( x = [x, \dot{x}, \theta, \dot{\theta}]^T \) the state vector of the cart–pendulum system. After setting \( F = 0 \), simulate the model by starting from the following initial conditions (initial state)

(a) \( x(0) = [0, 0, 0, 0]^T \)  (b) \( x(0) = [1, 0, 0, 0]^T \)  (c) \( x(0) = [0, 0, \pi/8, 0]^T \)

(d) \( x(0) = [0, 0, \pi, 0]^T \)  (e) \( x(0) = [1, 0, \pi, 0]^T \)  (f) \( x(0) = [0, 0, 7\pi/8, 0]^T \)  (g) \( x(0) = [1, 0.1, \pi, 0]^T \)

The initial state \( x(0) \) can be imposed by setting the initial condition of every integrator \((\text{Integrator} \rightarrow \text{Initial condition})\) to the appropriate value. From the (a)–(c) tests, deduce that any state of the type \( x_e = [\cdot, 0, 0, 0]^T \) \((\text{upward vertical equilibrium})\) is an unstable equilibrium state; conversely, from the (d)–(g) tests, deduce that any state of the type \( x_e = [\cdot, 0, \pi, 0]^T \) \((\text{downward vertical equilibrium})\) is an stable equilibrium state, but not asymptotically stable.

3. Assume that \( F = -kx \), which models a restoring spring of stiffness \( k \) connected to ground. Set \( k = 10 \).

Repeat the (a)–(c) tests of point 2, and deduce that \( x_e = [x, 0, 0, 0]^T \) with \( x \neq 0 \) is no longer an equilibrium state.

Repeat the (d)–(g) tests of point 2, with a spring stiffness \( k = 10 \). Deduce that \( x_e = [x, 0, \pi, 0]^T \) with \( x \neq 0 \) is no longer an equilibrium state; moreover, the state \( x_e = [0, 0, \pi, 0]^T \) is now an asymptotically stable equilibrium state.

4. Assume \( F = -kx \) with \( k = 10 \) as in point 3, and set \( b = 0 \) \((\text{i.e. system without energy dissipation})\).

Repeat the (d)–(f) tests of point 2, and deduce that the equilibrium state \( x_e = [0, 0, \pi, 0]^T \) is a stable equilibrium, but not asymptotically stable.

**Exercise 2: model linearisation and stability**

Reconsider the nonlinear dynamical model (1)–(2) of the cart–pendulum system. The linearised model about any downward equilibrium state \( x_e = [\cdot, 0, 0, 0]^T \) is:

\[
\Sigma_d : \begin{cases}
I_c \ddot{\theta} + ml \ddot{x} + mg \ell \theta = 0 \\
ml \ddot{\theta} + M_c \ddot{x} + bx = F
\end{cases}
\]

Conversely, the linearised model about any upward equilibrium state \( x_e = [\cdot, 0, 0, 0]^T \) is:

\[
\Sigma_u : \begin{cases}
I_c \ddot{\theta} - ml \ddot{x} - mg \ell \theta = 0 \\
-ml \ddot{\theta} + M_c \ddot{x} + bx = F
\end{cases}
\]

1. Derive a state–space representation \((A_d, B_d, C_d, D_d)\) of \( \Sigma_d \), with state vector \( x = [x, \dot{x}, \theta, \dot{\theta}]^T \), input \( F \) and outputs \( x \) and \( \theta \). Then, from the state–space representation, derive the two transfer functions \( P_{d, \theta}(s) \) and \( P_{d,x}(s) \) from the same input \( F \) to the outputs \( \theta \) and \( x \), respectively.

Similarly, derive a state–space representation \((A_u, B_u, C_u, D_u)\) of \( \Sigma_u \) with same state, input and output variables, and then the two transfer functions \( P_{u, \theta}(s) \) and \( P_{u,x}(s) \) (with inputs and outputs as specified above).

2. Compute the eigenvalues of \( A_d \), and plot them on the complex plane. Deduce the stability of the linearised model \( \Sigma_d \) from the location of the eigenvalues of \( A_d \) on the complex plane. By looking at these eigenvalues, what can be said about the stability of the original nonlinear model \( \Sigma_{nl} \) about the downward vertical equilibrium?

Repeat the analysis for the model \( \Sigma_u \). Is the linearised model stable? Does the behaviour of \( \Sigma_u \) allow to determine the instability of \( \Sigma_{nl} \) about the upward vertical equilibrium?

3. Discuss the bounded–input, bounded–output (BIBO) stability of the two linearised models \( \Sigma_d \) and \( \Sigma_u \) from the input \( F \) to the output \( \theta \), by computing the poles of the two transfer functions \( P_{d, \theta}(s) \) and \( P_{u, \theta}(s) \). Repeat the analysis for the output \( x \).

4. Compare the natural responses of the two models \( \Sigma_{nl} \) and \( \Sigma_u \) for the same initial state \( x(0) = [0, 0, \pi/8, 0]^T \).

Similarly, compare the natural responses of the two models \( \Sigma_{nl} \) and \( \Sigma_d \) for the same initial state \( x(0) = [0, 0, 7\pi/8, 0]^T \).
5. Let $\Sigma_d'$ denote the linearised model $\Sigma_d$ obtained by setting the force input equal to $F = -kx$ with $k = 10$ (similarly to what done in point 3 of exercise 1). Note that the force $F$ can be rewritten in terms of the state vector $x = [x, \dot{x}, \theta, \dot{\theta}]^T$ as follows

$$F = -Kx \quad \text{with} \quad K = [k, 0, 0, 0]$$  

Verify that a possible state–space representation for $\Sigma_d'$ is

$$\Sigma_d' : \begin{pmatrix} A_d', B_d', C_d', D_d' \end{pmatrix} = \begin{pmatrix} A_d - B_d K, 0, C_d, 0 \end{pmatrix}$$  

From the eigenvalues of $A_d'$, deduce the asymptotic stability of the model $\Sigma_d'$. What can be said about the stability of the nonlinear model $\Sigma_{nl}$ (with force input as specified above) about the downward vertical equilibrium $x_e = [0, 0, \pi, 0]^T$?

Exercise 3: simulation of a state feedback regulator

Suppose that the actuation force $F$ in (1)–(2) is generated, at any instant, as the following linear combination of the state variables:

$$F = -(k_1x + k_2\dot{x} + k_3\theta + k_4\dot{\theta})$$  

with $K = [k_1, k_2, k_3, k_4]$  

Equation (11) is the typical expression of a full state–feedback control law, with $K$ representing the state feedback (matrix) gain. In the following, assume $K = [-0.02, -1.1, 7.5, 0.7]$.

1. Test the state–feedback control law (11) on the Simulink model of the cart–pendulum system implemented in point 1 of exercise 1. Do the tests with the following initial conditions:

(a) $x(0) = [0, 0, \pi/8, 0]^T$  
(b) $x(0) = [0, 0, \pi/4, 0]^T$  
(c) $x(0) = [10, 0, 0, 0]^T$  
(d) $x(0) = [100, 0, 0, 0]^T$

Deduce that the proposed state feedback control law is capable of stabilising the nonlinear system $\Sigma_{nl}$ around its unstable upward vertical equilibrium state $x(0) = [0, 0, 0, 0]^T$, provided that the state perturbations are small (cases (a) and (c)).

2. Let $\Sigma_u'$ denote the linearised model $\Sigma_u$ obtained by setting the input force $F$ equal to (11). Verify that a possible state–space representation for $\Sigma_u'$ is

$$\Sigma_u' : \begin{pmatrix} A_u', B_u', C_u', D_u' \end{pmatrix} = \begin{pmatrix} A_u - B_u K, 0, C_u, 0 \end{pmatrix}$$  

From the eigenvalues of $A_u'$, deduce the asymptotic stability of the model $\Sigma_u'$. What can be said about the stability of the nonlinear model $\Sigma_{nl}$ (with force input as specified above) about the upward vertical equilibrium $x_e = [0, 0, 0, 0]^T$?

3. Assume that the force input is equal to

$$F = -Kx + F_d$$  

where $F_d$ is an extra input representing a force disturbance. Assume that $F_d$ is a pulse disturbance with amplitude=1 N and duration=0.1 s, and applied at $t = 20$s. Simulate the response of the nonlinear cart–pendulum model $\Sigma_{nl}$ with the force input (13) and initial state $x(0) = [0, 0, \pi/8, 0]^T$. Verify that the cart–pendulum system remains in its upward vertical position even in presence of the disturbance $F_d$. 

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# Self-evaluation

## Exercise 1

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