Laboratory activity 3: Friction and inertia estimation for a DC servomotor

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1 Activity goal

The goal of this laboratory activity is to estimate the friction and inertia parameters of the DC servomotor available in laboratory from data obtained with simple experimental tests. The parameters will be used in the next laboratory activity to improve the performance of the PI speed controller, in particular by implementing a feedforward inertia plus friction compensation scheme.

2 Estimation of model parameters

In principle, the control design can be carried out on a nominal model of the plant, provided that a sufficiently large phase margin is selected for tolerating possible parameter uncertainties that could potentially destabilise the control system. However, better performances can be usually achieved when the actual values of the plant parameters are estimated from the experimental data. In this section it is described a simple experimental procedure for the estimation of the main mechanical parameters, such as the friction (static and viscous) and the total rotor inertia. These parameters can be used to improve the control design, either by redesigning the feedback controller with the updated model plant, or by implementing a feedforward compensator through which better tracking performance can be achieved.

2.1 Friction estimation

Reconsider the overall mechanical dynamics at motor side derived in the handout of the first laboratory activity, and reported below for reader’s convenience

\[ J_{eq} \frac{d\omega_m}{dt} + B_{eq} \omega_m = \tau_m - \frac{1}{N} \tau_d \]

(overall mechanical dynamics at motor side)

where

\[ J_{eq} = J_m + \frac{J_l}{N^2}, \quad B_{eq} = B_m + \frac{B_l}{N^2} \]

and

\[ \tau_d = \tau_{sf} \text{sign(}\omega_l) , \quad \tau_{sf} > 0 \quad \text{(Coulomb friction at load side)} \]

After rearranging terms, the following torque balance equation results

\[ \tau_m = J_{eq} \frac{d\omega_m}{dt} + B_{eq} \omega_m + \frac{1}{N} \tau_d \]
which shows that at any instant, a fraction of the motor torque, namely the inertial torque component

\[
\tau_i \triangleq J_{eq} \frac{d\omega_m}{dt}
\]  

is used to accelerate the equivalent rotor inertia \( J_{eq} \), while the remaining part is required to overcome the total friction torque

\[
\tau'_f = B_{eq} \omega_m + \frac{1}{N} \tau_d 
\]  

(total friction at motor side)

Note that the total friction torque consists of two terms, namely the viscous friction torque component \( B_{eq} \omega_m \), and the static (or Coulomb) friction component \( \tau_d / N \) (as seen at motor side).

Suppose that the motor is operating at steady state with constant speed, i.e. \( d\omega_m/dt = 0 \); then from the torque balance equation (4), it follows that

\[
\tau_m = \tau'_f = B_{eq} \omega_m + \frac{\tau_{sf}}{N} \text{sign}(\omega_m) \quad \text{with} \quad \frac{d\omega_m}{dt} = 0
\]  

which shows that for keeping the motor running at constant speed, the motor torque \( \tau_m \) has to exactly balance the friction torque \( \tau'_f \). This torque balance can be effectively exploited to estimate the two friction parameters \( B_{eq} \) and \( \tau_{sf} \) in (7), as shown in the remaining part of this section. In fact, suppose to measure the motor torque at different constant speed levels. A direct measurement is obviously not possible with the available experimental setup; however, the motor law

\[
\tau_m = k_t i_a 
\]  

(law of motors)

shows that an indirect measurement is possible by sensing the motor current \( i_a \), provided that the torque constant \( K_t \) is known. In the experimental setup, remind that the armature current can be indirectly measured by sensing the voltage drop across the shunt resistor \( R_s \). Once the torque is known, the viscous and static friction coefficients in the affine function (7) can be estimated by performing a conventional linear least squares (LS) fitting of the torque vs speed data. For such purpose, rewrite the model (7) in linear regression form

\[
\tau_m = \tau'_f = \varphi^T \theta
\]  

where

\[
\varphi^T = \left[ \omega_m, \ (1/N) \text{sign}(\omega_m) \right], \quad \theta = \left[ B_{eq}, \ \tau_{sf} \right]^T
\]  

are, respectively, the vectors of regressors and unknown parameters to be estimated. Let

\[
Z^M = \{\tau_{m,k}, \ \omega_{m,k}\} \quad \text{with} \quad k = 0, \ldots, M - 1
\]  

denote the set of torque/speed pairs measured at \( M \) different constant speed levels. The vector of unknown parameters \( \theta \) can be determined by minimising the quadratic cost function (quadratic error)

\[
J = \sum_{k=0}^{M-1} \left( \tau_{m,k} - \varphi_k^T \theta \right)^2
\]  

\footnote{Alternatively, at load side, the total friction torque is equal to: \( \tau_f = N \tau'_f = N^2 B_{eq} \omega_l + \tau_d \)}
where
\[ \varphi^T_k = [\omega_{m,k}, (1/N) \text{sign}(\omega_{m,k})] \] (13)

With the notation
\[ Y = \begin{bmatrix} \tau_{m,0} \\ \tau_{m,1} \\ \vdots \\ \tau_{m,M-1} \end{bmatrix} \in \mathbb{R}^{M \times 1}, \quad \Phi = \begin{bmatrix} \varphi_0^T \\ \varphi_1^T \\ \vdots \\ \varphi_{M-1}^T \end{bmatrix} \in \mathbb{R}^{M \times 2} \] (14)

the cost function can be rewritten as
\[ J = (Y - \Phi \theta)^T (Y - \Phi \theta) \] (15)

which is minimised by the least squares (LS) solution
\[ \hat{\theta}_{LS} = (\Phi^T \Phi)^{-1} \Phi^T Y \] (16)

The LS estimate \( \hat{\tau}'_f \) of the total friction torque is therefore
\[ \hat{\tau}'_f = \hat{B}_{eq} \omega_m + \frac{\tau_{sf}}{N} \text{sign}(\omega_m) \] (17)

Note: in Matlab, suppose to have defined the matrices \( \Phi \) and \( Y \) with variables names Phi and Y; then, the least squares solution (16) can be computed by using the left matrix division operator “\”, i.e. \( \text{thLS} = \text{Phi}\backslash\text{Y} \) (consult \text{mldivide} on the online help).

2.2 Inertia estimation

Suppose to impose a constant acceleration/deceleration to the total gearmotor inertia, i.e. to increase/decrease the motor speed at a constant rate. From (4)–(6) it follows that an estimate \( \hat{\tau}_i \) of the inertial torque component \( \tau_i \) used to accelerate/decelerate the equivalent inertia can be obtained by subtracting from the measured motor torque \( \tau_m \) the friction torque \( \hat{\tau}'_f \) estimated with the procedure outlined in the previous Sec. 2.1, namely
\[ \hat{\tau}_i = \tau_m - \hat{\tau}'_f = \hat{J}_{eq} \frac{d\omega_m}{dt} \quad \text{with} \quad \frac{d\omega_m}{dt} \neq 0 \] (18)

Let \( \hat{\tau}_i^+ < 0 \) be the average inertial torque estimated with constant acceleration \( a_{m+} = d\omega_m/dt > 0 \); similarly, let \( \hat{\tau}_i^- > 0 \) be the average inertial torque estimated with constant deceleration \( a_{m-} = d\omega_m/dt < 0 \). Then, the total rotor inertia can be estimated as follows
\[ \hat{J}_{eq} = \frac{\hat{\tau}_i^+ - \hat{\tau}_i^-}{a_{m+} - a_{m-}} \] (19)

If multiple acceleration/deceleration phases are repeated over time, then the estimate of the total rotor inertia can be obtained by averaging the single inertia estimates obtained for each acceleration/deceleration phase, i.e.
\[ \hat{J}_{eq} = \frac{1}{M} \sum_{n=1}^{M} \hat{J}_{eq,n} \] (20)
where $J_{eq,n}$ is the inertia estimate obtained for the $n$th acceleration/deceleration phase (using (19)), namely

$$J_{eq,n} = \frac{\hat{\tau}_{i+n} - \hat{\tau}_{i-n}}{a_{m+n} - a_{m-n}}$$  \hspace{1cm} (21)

and $M$ denotes the total number of phases.

### 2.3 Combined friction and inertia estimation

A single experimental test can be used as a replacement of the previous two tests for the simultaneous estimation of the friction and inertia parameters. Rewrite the torque balance (4) in linear regression form

$$\tau_m = \varphi^T \theta$$  \hspace{1cm} (22)

where

$$\varphi^T = \left[ \frac{d\omega_m}{dt}, \omega_m, \left(1/N\right) \text{sign}(\omega_m) \right]$$  \hspace{1cm} \theta = \left[ J_{eq}, B_{eq}, \tau_{sf} \right]^T$$  \hspace{1cm} (23)

are, respectively, the vectors of regressors and unknown parameters to be estimated. The estimation problem can be formulated as follows: with the availability of the experimental data (measurements)

$$Z^M = \{\tau_m(t_k), \omega_m(t_k)\} \quad \text{with} \quad t_k = kT_s, \quad k = 0, \ldots, M - 1$$  \hspace{1cm} (24)

determine the vector of parameters $\theta$ that minimises the quadratic cost function (quadratic error)

$$J = \sum_{k=0}^{M-1} \left[ \tau_m(t_k) - \varphi^T(t_k) \theta \right]^2$$  \hspace{1cm} (25)

With the notation

$$Y = \begin{bmatrix} \tau_m(t_0) \\ \tau_m(t_1) \\ \vdots \\ \tau_m(t_{M-1}) \end{bmatrix} \in \mathbb{R}^{M \times 1}, \quad \Phi = \begin{bmatrix} \varphi^T(t_0) \\ \varphi^T(t_1) \\ \vdots \\ \varphi^T(t_{M-1}) \end{bmatrix} \in \mathbb{R}^{M \times 3}$$  \hspace{1cm} (26)

the cost function can be rewritten as in (15), which is then minimised by the LS solution

$$\hat{\theta}_{LS} = \begin{bmatrix} \hat{J}_{eq} \\ \hat{B}_{eq} \\ \hat{\tau}_{sf} \end{bmatrix}$$  \hspace{1cm} (27)

which is computed by using the expression (16). For the identifiability of the model parameters, the data (24) must be collected under sufficient excitation conditions of the plant dynamics. For such purpose, it is sufficient to impose constant acceleration/deceleration phases to the motor during the data collection experiment.
3 Laboratory assignments: post–processing of experimental data

3.1 Estimation of friction parameters

1. Consider the experimental data collected in the laboratory activity 2 using the staircase speed reference

\[ \omega^*_{l}(t) = k \Delta \omega \quad \text{with} \quad t \in [(k - 1) \Delta T, k \Delta T), \quad k = 1, \cdots, N^* \quad (28) \]

Note that the measured motor current is very noisy, due to the commutation of the brushes contacts occurring while the motor is rotating. To reduce the commutation noise, the current measurement can be filtered with a suitable low–pass filter, before proceeding with the estimation of the friction parameters. A second order Butterworth low–pass filter with 20 Hz cut–off frequency, namely the filter

\[ H_i(s) = \frac{\omega_{c,i}^2}{s^2 + 2 \delta_i \omega_{c,i} s + \omega_{c,i}^2} \quad \text{with} \quad \omega_{c,i} = 2\pi 20, \quad \delta_i = 1/\sqrt{2} \quad (29) \]

is sufficient for the purpose. Since all the experimental data are collected with a fixed sampling time equal to \( T_s = 1 \text{ms} \), data filtering should be performed in the discrete–time domain with a discrete–time filter. Therefore, after defining in Matlab the continuous–time filter (29), use the CST routine c2d to discretise the filter with a suitable method (e.g. ‘tustin’); then, use the Matlab routine filter to compute the filtered version of the the motor current measure. The Matlab excerpt for the motor current filtering is reported below:

```matlab
1 % low-pass filtering of armature current measurement
2 wc_i = 2*pi*20;
3 di = 1/sqrt(2);
4 sysHi = tf(wc_i^2, [1, 2*di*wc_i, wc_i^2]);
5 sysHi = c2d(sysHi, sim.Ts, 'tustin');
6 [numHi, denHi] = tfdata(sysHi, 'v');
7 ia = filter(numHi, denHi, ia);
```

Notes:

– in line 4, the LTI object sysHi for the continuous–time filter (29) is defined.
– in line 5, a discrete equivalent (using the Tustin’s method) of the filter (29) is obtained.
– in line 7, the motor current signal ia is filtered by using the Matlab routine filter.
– an alternative to the lines 2–6 would be to use the routine butter of the Signal Processing Toolbox, which defines the numerator/denominator of a discrete–time low–pass Butterworth filter of specified order and cut–off frequency.

2. For each time interval

\[ t \in [(k - 1) \Delta T, k \Delta T), \quad k = 1, \cdots, N^* \quad (30) \]

within which the motor speed is constant, compute the average values \( \omega_{m,k} = N \omega_{l,k} \) and \( \tau_{m,k} = K_l i_{a,k} \) of the motor speed and torque. In each time interval, avoid to consider the
initial speed and torque transients in the computation of the average values (i.e. consider only the portion of data where a constant steady state value is reached – see shaded areas in Fig. 1a).

Remind that two tests were performed, one with a positive staircase speed reference, and the other with a negative reference. Consider to compute the speed/torque averages for the data of both tests. Denote with $Z^N_+ = \{ \tau_{m,k}, \omega_{m,k} \}$ the data set obtained with the positive reference speed, and with $Z^N_- = \{ \tau_{m,k}, \omega_{m,k} \}$ the data set obtained with the negative reference speed.

3. With the data set $Z^N_+ = \{ \tau_{m,k}, \omega_{m,k} \}$ obtained in point 2, obtain the friction parameters $\hat{B}_{eq+}$ and $\hat{\tau}_{sf+}$ using the estimation procedure outlined in Sec. 2.1. Please pay attention to use the appropriate measurement units when applying the estimation procedure: in particular, the torque should be expressed in [Nm], and the speed in $[\text{rad/s}]$. In this way, $\hat{B}_{eq+}$ will be expressed in $[\text{Nm}/(\text{rad/s})]$, and $\hat{\tau}_{sf+}$ in [Nm].

Repeat the estimation for the data set $Z^N_-$, obtaining the estimates $\hat{B}_{eq-}$ and $\hat{\tau}_{sf-}$. Then, obtain the overall estimates of the two friction parameters as follows

$$\hat{B}_{eq} = \frac{\hat{B}_{eq+} + \hat{B}_{eq-}}{2}, \quad \hat{\tau}_{sf} = \frac{\hat{\tau}_{sf+} + \hat{\tau}_{sf-}}{2}$$

(31)

The final friction torque characteristic (motor side, with speed plotted in [krpm]) should look like that reported in Fig. 1b.
3.2 Estimation of inertia parameters

1. Consider the experimental data collected in the laboratory activity 2 using the triangular wave speed reference (i.e. speed reference with constant acceleration/deceleration phases)

\[ \dot{\omega}^*(t) = \int_0^t a^*_l(\tau) \, d\tau \quad (32) \]

with

\[ a^*_l(t) = (-1)^k A \quad \text{for} \quad t \in [ k \Delta T, (k+1)\Delta T ) , \quad k = 0, \cdots, N^* - 1 \quad (33) \]

As done in point 1 of Sec. 3.1, use the filter (29) to remove the brushes commutation noise in the current measurement (see bottom plot in Fig. 2a).

2. For estimating the equivalent inertia with (19), the motor acceleration is required. This can be obtained by differentiation of the motor speed with a suitable high–pass filter (see top plot in Fig. 2b). The filter cut–off frequency and roll–off must be chosen to avoid excessive amplification of the encoder quantisation noise at high frequency. A second order high–pass filter of the type

\[ H_a(s) = \frac{\omega_{c,a}^2 s}{s^2 + 2\delta_a \omega_{c,a} s + \omega_{c,a}^2} \quad \text{with} \quad \omega_{c,a} = 2\pi 20, \quad \delta_a = 1/\sqrt{2} \quad (34) \]

is sufficient for the purpose. As for the current measurement, the speed filtering should be performed in the discrete–time domain (indeed, the speed measurement is a discrete–time signal), after discretising (34) with a suitable method. The Matlab excerpt for computing the motor acceleration from the speed measurement (both considered at load side) is reported below:

```matlab
1  % high–pass filtering of speed measurement (to get the acceleration)
2  wca = 2*pi*20;
3  da = 1/sqrt(2);
4  sysFa = tf([0, wca^2, 0], [1, 2*da*wca, wca^2]);
5  sysFa = c2d(sysFa, sim.Ts, 'tustin');
6  [numFa, denFa] = tfdata(sysFa, 'v');
7  al = filter(numFa, denFa, wl);
```

3. Use the estimated friction parameters (31) of Sec. 3.1 to compute the friction torque signal

\[ \hat{\tau}_f(t_k) = \hat{B}_{eq} \omega_m(t_k) + \frac{\hat{\tau}_f}{N} \, \text{sign} \left( \omega_m(t_k) \right) \quad \text{with} \quad t_k = k \, T_s, \quad k = 0, \cdots, M-1 \quad (35) \]

where \( \omega_m(t_k) \) is the motor speed measured at the generic sampling time instant \( t_k \). Then, use (18) to compute the torque signal

\[ \hat{\tau}_i(t_k) = \tau_m(t_k) - \hat{\tau}_f(t_k) \quad (36) \]

which represent the fraction of motor torque used to accelerate the total inertia \( \hat{J}_{eq} \) (inertial torque component – see bottom plot in Fig. 2b). In (36), \( \tau_m(t_k) \) is the motor torque measured
4. For each time interval

$$t \in [k \Delta T, (k+1)\Delta T), \quad k = 0, \cdots, N^* - 1$$

within which the motor acceleration is constant, compute the average values $a_{m,k} = N a_{l,k}$ and $\hat{\tau}_{i,k}$ of the motor acceleration and inertial torque component. In each time interval, avoid to consider the initial acceleration and torque transients in the computation of the average values (see shaded areas in Fig. 2b).

Note that $a_{m,k}$ is a positive acceleration when $k$ is even, and negative otherwise.

5. Obtain the estimate of the equivalent inertia using the estimation procedure outlined in Sec. 2.2. In particular, the application of (20) yields

$$\hat{J}_{eq} = \frac{1}{P} \sum_{n=0}^{P-1} \frac{\hat{\tau}_{t,2n} - \hat{\tau}_{t,2n+1}}{a_{m,2n} - a_{m,2n+1}}$$

with $P = N^*/2$ (assuming $N^*$ even). In (38) it has been used the fact that given the acceleration reference (33), the average value $a_{m,k}$ of the motor acceleration obtained in point 4 is positive when $k$ is even, and negative otherwise.