INP3060357 – Channel coding – Year 2016/2017

Homework 2

Solve (at least) 2 exercises at your choice, where at least one should be taken from the “message passing” set 1-6, and at least one should be taken from the “entropy and capacity” set 7-12. The class should be so kind to make sure that at least one solution for each exercise is available.

The homework is due by 12:00 of January 11, 2017.

Message passing

Exercice 1 (Message passing example) Consider function

\[ f(x_1, x_2, x_3) = \frac{1}{K} g(x_1) g(x_2) g(1 - x_3) \delta_{x_1, x_2} \delta_{x_2, 1 - x_3} \]

where \( x_i \) are binary variables, \( g(x) = 0.1 \) for \( x = 0 \) and \( 0.9 \) for \( x = 1 \), \( K \) is a normalization factor to have \( \sum_{x} f(x) = 1 \). It is required to:

1. evaluate \( K \),
2. give a graphical representation to \( f \),
3. apply (step by step) message passing to marginalize with respect to each \( x_i \).

Exercice 2 (Min sum algorithm for convolutional codes) Describe in detail the min-sum-algorithm when applied to a binary convolutional code, by specifying the message passing algorithm update for the three cases of interest, namely forward, backward, and extrinsic information messages. What about its formulation using LLRs? What is the relation with the Viterbi algorithm?

Exercice 3 (Min sum algorithm for LDPCs) Describe in detail the min-sum-algorithm when applied to a LDPC, by specifying the message updates at variable and check nodes. What about its formulation using LLRs? Try making the LLR formulation as compact as you possibly can, in which case you should obtain (244) at p. 150 of the lecture notes.

Exercice 4 (Non linear encoding) In order to enhance the strength of a binary code, non-linear elements can be inserted in the encoding procedure. Consider the non-linear element

\[
\begin{align*}
[u_0, u_1] & \rightarrow [c_0c_1c_2c_3] \\
[00] & \rightarrow [0000] \\
[01] & \rightarrow [1000] \\
[10] & \rightarrow [0100] \\
\end{align*}
\]

in a binary coding context. Identify the sum-product and min-sum message passing rules that use a LLR mapping. Try giving them a compact representation.
Exercice 5 (Decoding of a serial concatenated code) Consider a serial concatenated code defined as follows:

1. outer convolutional code (rate $\frac{1}{2}$),
2. bit interleaver (random interleaver),
3. bit to $q$-ary symbol map ($q = 2^3 = 8$);
4. inner differential encoding ($y_k = y_{k-1} + x_k \mod 8$),
5. 8-PSK Gray mapping.

This is a form of RA code which is typical of, e.g., PLC-G3 and PLC-PRIME standards. It is required to:

1. give a graphical representation to the code;
2. identify the specific message passing algorithm;
3. propose an initialization and a schedule for the message passing algorithm.

Exercice 6 (EXIT Charts for irregular LDPCs) Consider an irregular LDPC code of rate $\frac{1}{2}$, with associated functions $\lambda(\cdot)$ and $\rho(\cdot)$. Identify the property that functions $\lambda$ and $\rho$ (or, better, their coefficients) must satisfy in order to ensure that the resulting code rate is $R = \frac{1}{2}$. Propose two functions $\lambda(\cdot)$ and $\rho(\cdot)$ that guarantee a limit probability $p^*$ higher than that of a regular LDPC code with $d_\lambda = 3$ and $d_\rho = 6$, and rate $\frac{1}{2}$. Graphically illustrate the result on an EXIT chart.

Entropy and capacity

Exercice 7 (Properties of entropy and mutual information) Prove the following and find conditions for equality:

1. $0 \leq H(x|y) \leq H(x)$ (information reduces entropy);
2. $0 \leq I(x;y) \leq \min(H(x), H(y))$;
3. $\max(H(x), H(y)) \leq H(x, y) \leq H(x) + H(y)$;
4. $H(x_1, \ldots, x_N) \leq H(x_1) + H(x_2) + \ldots + H(x_N)$.

Exercice 8 (Differential entropy limits) Prove that for a continuous random variable $x$ with $0 \leq x \leq a$, the differential entropy $H(x)$ satisfies $H(x) \leq \log_2 a$ with equality if and only if $x$ is uniformly distributed.

Exercice 9 (Capacity of the binary erasure channel) Derive the capacity for the BEC.

Exercice 10 (Binary PAM under hard decoding) Consider binary PAM with hard decoding under reliable communications, whose spectral efficiency satisfies $\frac{1}{2} \rho < 1 - h(Q(\sqrt{T}))$, where $h(x)$ is the entropy function of a binary random variable. Draw the limit in MatLab versus $E_b/N_0$, and prove analytically that the minimum value for energy efficiency is $E_b/N_0 = \frac{\pi}{2} \log_2 2$. 

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Exercice 11 (M-ary PAM under soft decoding) Consider the expression of information rate $I(x; y)$ in the case of M-ary PAM with soft decoding in AWGN.

1. Devise a MatLab program to draw $I(x; y)$ as a function of the SNR $\Gamma$ given the constellation size $M$ and the input probability mass distribution $p_x(a)$ stored in a vector. Hint: integration in MatLab can be performed numerically by use of the “quad” function; inline function definitions, e.g., “f = @(x,y) x.*y;” for $f(x, y) = x \cdot y$, may also be of help.

2. Plot information rate as a function of $\Gamma$ in case of equally likely input probabilities to obtain the plots seen during lectures.

3. Force a Gaussian-like distribution on $p_x(a)$ and plot information rates for some values of $M$. Do you see any improvement in information rate? Is this improvement general or relative to a limited region? What design suggestions can be deduced from this experiment?

Exercice 12 (M-ary PSK under soft decoding) Consider the expression of information rate $I(x; y)$ in the case of M-ary PSK with soft decoding in AWGN.

1. Devise a MatLab program to draw $I(x; y)$ as a function of the SNR $\Gamma$ given the constellation size $M$ and the input probability mass distribution $p_x(a)$ stored in a vector. Hint: 2D integration in MatLab can be performed numerically by use of the “quad2d” function; inline function definitions, e.g., “f = @(x,y) x.*y;” for $f(x, y) = x \cdot y$, may also be of help.

2. Plot information rate as a function of $\Gamma$ in case of equally likely input probabilities to obtain the plots seen during lectures.